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**SOME EXACT SOLUTIONS OF STEADY PLANE
NEWTONIAN, NON-NEWTONIAN AND MHD FLUID FLOWS**

by

FOTINI LABROPULU

**A Dissertation
submitted to the Faculty of Graduate Studies and Research
through the Department of Mathematics and Statistics
in partial fulfillment of the requirements for the
degree of Doctor of Philosophy at
the University of Windsor**

**Windsor , Ontario , Canada
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ABSTRACT

This dissertation is devoted to (i) an analytical investigation of steady plane magnetohydrodynamic (MHD) fluid flows and (ii) a numerical study of a viscoelastic second grade fluid flow.

In the analytical study, we pose and answer the following two questions for steady plane flow:

(a) Given a family of plane curves $l(x, y) = \text{constant}$, can a viscous fluid of constant viscosity flow along these curves?

(b) Given a family of streamlines in a viscous fluid flow of constant viscosity, what is the exact integral or exact solution of the flow defined by the given streamline pattern?

To answer these two questions, we develop a new approach using different curvilinear coordinate systems depending upon the problem under consideration. This approach has been used to recover some existing exact solutions and yield several new exact solutions of infinitely conducting MHD aligned, finitely conducting MHD aligned, infinitely conducting MHD orthogonal, infinitely conducting MHD variably-inclined and non-MHD fluid flows. In the case of MHD aligned fluid flow, some boundary value problems have been considered. We also show that the Harnel's problem for infinitely conducting MHD aligned fluid flow has more solutions than the four well-known solutions of the ordinary viscous fluid flow.

A study of confluent flows for MHD transverse-aligned flow is also carried out in this part of the analytical investigation. A fluid flow is defined to be confluent if two physically important families of curves coincide in the physical plane.

In the numerical part of this dissertation, we study the oblique flow of a viscoelastic second grade fluid impinging on a flat porous wall with suction or blowing on the wall. The behaviour of the fluid near the wall is investigated for various magnitudes of suction and blowing. We also investigate the effects of elasticity on this fluid flow.

**Respectfully Dedicated to My Beloved Mother
and to the Loving Memory of My Grandfather**

ACKNOWLEDGEMENT

I would like to acknowledge the help I have received from many people throughout my studies. First of all, I wish to express my most sincere thanks and appreciation to my supervisor, Dr. O.P. Chandna for his many valuable ideas, capable guidance and consideration throughout the course of this research. I would like to take this opportunity to express my heartfelt gratitude to him for all his help and sincere friendship in all my years here. I shall always be deeply indebted to him.

I also wish to express my thanks to Dr. R.J. Caron, Chairman of the Department of Mathematics and Statistics for all his help and support and also for providing the computer facilities for the production of this dissertation.

A special thanks to Dr. J. M. Doreepaal for his valuable assistance without which a significant part of this dissertation would not have been possible.

I am very grateful to my family for their support and encouragement all through the years of my studies.

This research was supported by Dr. O.P. Chandna's NSERC grant, graduate assistantships and several scholarships from the University of Windsor as well as the Ontario Graduate Scholarship. I am deeply indebted for this support.

Many thanks to my external examiner Dr. L. Debnath for taking the time to examine this work and for all of his valuable suggestions.

I also would like to thank the members of the examining committee Drs. P.N. Kaloni, J.A. McCorquodale, R.M. Barron and N.G. Zamani for all valuable criticisms and suggestions.

My heartfelt gratitude to my fiance Mr. Iqbal Husain for all his help, encourage-

ment and support throughout my studies.

I wish to express a special thanks to a dear friend Mrs. Ella May Bunt for her most sincere support and friendship. Her presence and encouragement have been unfailing throughout my studies.

To Dr. A.C. Smith, Dr. K.L. Duggal, Dr. D. Tracy, Dr. F. Lemire, Dr. H.R. Atkinson and Dr. G. McPhail, thanks for their assistance.

Last but not least, thanks are also extended to my friends Mrs. A. Rowland, Mrs. R. Gignac, Mr. E. Oku-Ukpong and Mr. S. Venkatasubramanian for all their help and encouragement.

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NOMENCLATURE

<u>Symbol</u>	<u>Dimension^(*)</u>	<u>Meaning</u>
A_1	T^{-1}	1 st Rivlin-Ericksen tensor
A_2	T^{-2}	2 nd Rivlin-Ericksen tensor
H	$QL^{-1}T^{-1}$	magnetic field vector
H_1, H_2, H_3	$QL^{-1}T^{-1}$	cartesian components of magnetic field vector
h	$ML^{-1}T^{-2}$	generalized energy function
k	$ML^{-1}T^{-2}$	kinetic energy
I	-	unit tensor
p	$ML^{-1}T^{-2}$	pressure
r	L	radius, polar coordinate
T	$ML^{-1}T^{-2}$	stress tensor
u, v, w	LT^{-1}	cartesian components of velocity
V	LT^{-1}	velocity vector
We	-	Weissenberg number
x, y, z	L	cartesian coordinates
α_1, α_2	ML^{-1}	second order material coefficients
μ	$ML^{-1}T^{-1}$	viscosity
θ	-	angle, polar coordinate
ρ	ML^{-3}	density
σ	$M^{-1}L^{-3}TQ^2$	electrical conductivity

<u>Symbol</u>	<u>Dimension^(*)</u>	<u>Meaning</u>
τ	$ML^{-1}T^{-2}$	shear stress
ψ	L^2T^{-1}	streamfunction
ω	T^{-1}	vorticity function
Ω	$QL^{-2}T^{-1}$	current density
μ^*	MLQ^{-2}	magnetic permeability
α	-	angle

(*) Mass [M], Length [L], Time [T], Electric charge [Q]

The fundamental quantities of mechanics are: length [L], time [T], mass [M], electric charge [Q] and their corresponding units are meter (m), seconds (s), kilogramme (kg) and coulomb (C).

CHAPTER 1

INTRODUCTION

1.1. MAGNETOHYDRODYNAMICS.

Magnetofluidynamics (MFD) is the study of the macroscopic interactions of electrically conducting fluids in the presence of a magnetic field. The magnetic field influences the fluid motion and the fluid motion changes, in turn, the magnetic field. As would be expected, both the equations of fluid mechanics and of electromagnetism feature in the description of a magnetofluiddynamic flow: each imposing something of its own distinctive features on the subject.

Magnetohydrodynamics (MHD) is a subfield of MFD when the fluid is incompressible. However, the use of the term MHD for both compressible and incompressible fluid flows is quite common in literature.

When a conducting fluid moves through a magnetic field, an electric field and consequently a current may be induced and, in turn, the current interacts with the magnetic field to produce a body force on the fluid.

From the historical point of view, the first investigations in MFD were due to Faraday in 1836. Faraday attempted to detect the electric field induced by the motion of the conducting water of the Thames River in the presence of the earth's magnetic field. His experiments failed because of the low electrical conductivity of the fluid. A second important stage in the development of this science is represented

by the experiments of Hartman and Lazarus in 1937 who pointed out the magnetic field influence on the motion of the fluid. In their work, mercury flowing through a tube in the presence of a strong magnetic field was used as the conducting fluid. Although the experimental results confirmed the theoretical studies, no practical reason for continuing these investigations existed at that time.

The first genuine magnetohydrodynamic problems were posed by astrophysicists. In 1942, the Swedish scientist H. Alfvén proved that new waves, unknown to both fluid mechanics and electromagnetism can be propagated through a conducting fluid in a magnetic field.

Of more recent origin is the interest in MHD and MFD spurred by problems encountered in aeronautics and astronautics. At the flight velocities at which future aircraft and space vehicles are expected to operate, the energy levels involved are such as to cause substantial ionization in the atmospheric environment of the vehicles and/or to require propulsion systems capable of producing particle streams with directed energies much higher than those possible with present-day chemical fuels. Another recent application of this field is the use of constant intensity magnetic fields in metallurgy to direct flow, to stir, and to levitate conducting melts. In recent years magnetic fields have also been applied to damp convection in systems for the melt crystal growth of semiconductors and thereby control fluctuations in the composition of the crystal. The control of particle settling in metallic systems is another potential important application of magnetic fields.

The mathematical study of magnetohydrodynamics is concerned primarily with the partial differential equations which arise from the well known physical conservation laws. In general, this study involves a system of seven partial differential equations in seven unknown functions of which three are dynamic, three are magnetodynamic and one is thermodynamic. Because of the complexity of the governing

equations, most of the MHD investigations to date have been carried out with additional assumptions. Lundquist [1952] investigated unsteady flows of fluids having infinite electrical conductivity. Resler and McCune [1959] developed linearized MHD for inviscid fluids. McCune [1960] studied the flow past thin airfoils in fluids of finite electrical conductivity in the presence of an orthogonal applied magnetic field. The flow of a viscous electrically conducting fluid past a semi-infinite flat plate in the presence of an aligned magnetic field was considered by Greenspan [1961]. Various other MHD problems with additional assumptions were investigated by Chandrasekhar [1961], Pai [1962] and Ranger [1969]. The other approach that researchers have adopted to overcome the complexity of the subject is to isolate special classes of flows so that the existing fluid dynamic techniques may be applied to MHD problems. Based on the assumed geometric relationship between the velocity field vector and the magnetic field vector, researchers have classified special classes of MHD flows such as aligned flows, orthogonal flows, constantly-inclined flows, transverse flows, variably-inclined flows and transverse-aligned flows. Grad [1960] gave the idea of studying these special classes of flows which are also called reducible MHD flows. Definitions of reducible flows considered in this work along with some research work from the literature for these flows are given in the following:

Aligned Flow

An MHD flow is said to be *aligned* or *parallel* if the magnetic field vector and the velocity field vector are everywhere parallel to one another in the flow field. Steady aligned flows were one of the first flows studied. Chandrasekhar [1959] investigated the stability of an aligned flow problem for the case of inviscid incompressible fluids. Sears and Resler [1959] studied the steady, plane flow of an incompressible fluid of infinite electrical conductivity past thin bodies with two different orientations of

the applied magnetic field, parallel and perpendicular to the uniform velocity far away from the body. They showed that the streamlines and magnetic lines must be parallel to one another everywhere in the flow region when they are aligned at infinity. Stewartson [1959] considered the motion of a non-conducting body through a perfectly conducting fluid in an aligned magnetic field. Lary [1962] investigated the aligned flow past thin airfoils for arbitrary values of conductivity. Vinokur [1961], Peyret [1962], Smith [1963] and Kingston and Power [1968] obtained many results for aligned flows of a compressible fluid by extending methods of rotational gas flows. Chandna and Nath [1972] developed a substitution principle, for fluids having an arbitrary equation of state that corresponds to Prim's [1952] substitution principle for classical gas flows. Chandna, Murgai and Shridhar [1985] applied the hodograph method to investigate the geometries and the solutions when the velocity magnitude is constant on each individual streamline. Creeping flow of a conducting fluid past axisymmetric bodies in the presence of an aligned magnetic field was investigated by Kyrilidis, Brown and Walker [1990]. Chandna and Labropulu [1990] investigated confluent flows for finitely conducting aligned flows. Chandna, Husain and Labropulu [1991] obtained solutions for finitely conducting compressible aligned parallel flows. Chandna and Labropulu [1992] obtained solutions to Rabinovich type problems for aligned MHD flows.

Orthogonal Flow

An MHD flow is said to be *orthogonal* if the velocity and the magnetic field vectors are perpendicular to each other everywhere in the flow plane. Ladikov [1962] studied orthogonal flows of an inviscid compressible fluid having infinite electrical conductivity and obtained two Bernoulli equations for these flows. Power and Walker [1965] and Power and Talbot [1969] studied plane compressible orthogonal flows by reducing the problem to that of rotational gas dynamic flows. Kingston and

Talbot [1969] completely classified the possible flow configurations for the viscous incompressible fluid. Garg and Chandna [1976] applied Martin's method to study Hamel's problem for orthogonal steady plane viscous incompressible fluids of infinite electrical conductivity. Barron, Chandna and Prabakaran [1984] classified the flow geometries of orthogonal electro-magneto-fluid-dynamic flows. MHD orthogonal flows with straight streamlines and flows with streamlines as the involutes of a curve were investigated by Chandna, Barron and Chew [1989].

Variably-Inclined Flow

An MHD flow is said to be *variably-inclined* if the angle between the velocity vector and the magnetic vector is varying from point to point in the flow region. Chandna, Barron and Chew [1982, 1983] applied hodograph transformation and Martin's method respectively to study steady plane flows of viscous, incompressible and electrically conducting fluids having infinite electrical conductivity. Steady and unsteady hydromagnetic flows of a viscous incompressible and electrically conducting fluid in the presence of an inclined magnetic field in a rotating channel have been studied by Ghosh [1991]. Chandna, Labropulu and Husain [1991] obtained exact solutions of variably-inclined MHD parallel flows. The mathematical equations are complicated and not much work has been done for these flows.

Transverse-aligned Flow

An MHD flow is said to be *transverse-aligned* if the plane projection of the spatial magnetic field on the plane of flow is everywhere parallel to the planar velocity field. Grad [1960], Imai [1960, 1962] and Hasimoto [1959] studied these flows to investigate some of their basic characteristics. Labropulu and Chandna [1990] studied confluent flows of finitely conducting transverse-aligned flows.

1.2 EXACT SOLUTIONS.

The equations governing the flow of Newtonian and non-Newtonian fluid flows form systems of non-linear partial differential equations. These systems have no general solutions. Fascinated by various flows, many researchers have grappled with the analysis of these systems since their formulations. Their analyses, often inspired by physical intuition, have resulted in the discovery of many outstanding concepts in the theory of partial differential equations. Theory of characteristics, Green's function, stability analysis, shock waves and eigenfunctions (c.f. Myint-U and Debnath [1987]) are some of the many excellent achievements from researchers of fluid mechanics. Ames [1965] has given an excellent treatment of the various methods employed for solving these equations and their applications. However, the search for exact solutions of these nonlinear systems continues. The two definitions for an exact solution of an ordinary viscous incompressible flow problem are:

(i) A set of functions for the velocity components and the fluid pressure constitutes an exact solution of the flow equations if these equations are satisfied by these functions for all values of the independent variables, for all values of the fluid density ρ and the fluid viscosity μ .

(ii) A set of functions for the velocity components and the fluid pressure constitutes an exact solution or exact integral of the flow equations if these equations and the boundary conditions of a realistically imposed physical problem are satisfied for all values of the independent variables and for all values of ρ and μ .

According to the first definition, exact solutions of flow equations are found without considering any real physical problem. The determination of these solutions does not involve any solid or movable boundaries. However, these solutions provide the nature of permissible solutions of the system. An exact solution thus obtained often leads future researchers to its applications for a real problem. As an example,

Riabouchinsky's solutions of Navier-Stokes equations for unbounded domain were used by Stuart [1959], Tamada [1979] and Dorrepaal [1986] in their works of oblique stagnation point fluid flow.

In the other definition, one must also postulate appropriate initial and boundary conditions. Therefore, this research work involves the study of boundary value problems which result from coupling the boundary and initial conditions with the governing system of differential equations. These boundary value problems of fluid mechanics are exceedingly difficult and progress would have been negligible if rigorous mathematics had not been supplemented by various plausible intuitive hypotheses.

Exact solutions are important for the following two reasons as reported by Wang [1991]:

(1) The solutions represent fundamental fluid-dynamic flows. Also, owing to the uniform validity of exact solutions, the basic phenomena described by the flow equations can be more closely studied.

(2) The exact solutions serve as standards for checking the accuracies of the many approximate methods, whether they are numerical, asymptotic or empirical. Current advances in computer technology make the complete numerical integration of the flow equations more feasible. However, the accuracy of the results can be ascertained by a comparison with an exact solution.

The existing exact solutions have been published in a wide variety of journals, spanning a century or more. The only comprehensive review of exact solutions of Navier-Stokes equations is due to Berker [1963] who expanded on the earlier works of Berker [1936] and Dryden *et al.* [1932]. A most recent review of exact solutions is due to Wang [1991]. Exact solutions for ordinary compressible flows and MFD flows are rare and are well documented in texts by Courant and Friedrichs [1948]

for compressible flows and by Dragos [1975] for MFD flows.

Most of these exact solutions are obtained by a variety of methods. These methods consist of making certain hypotheses either on the form of the velocity field or the pressure function. The works of Taylor [1923], Nemenyi [1951], Jeffrey [1915], Riabouchinsky [1924], Kaloni and Huschilt [1984], Kaloni and Siddiqui [1986] and Hui [1987] are among those who have used these so-called indirect methods.

1.3 CONFLUENT FLOWS.

A steady plane fluid flow is said to be *confluent* if two of the physically important families of curves coincide in the physical plane. The fluid flow is said to be *fluent* otherwise. Well-known Prandtl-Meyer flows, when curves of constant speed are also curves of constant flow inclination, were extensively studied by Martin [1950]. Later Govindaraju [1981] studied confluent flows.

Martin [1971] proposed fifteen confluent flows for further study and gave a bar system for ordinary steady plane viscous incompressible fluid flow governed by the Navier-Stokes equations. The fifteen confluent flows he proposed are the flows when any two of the following six families of curves coincide in the physical plane:

- (a) streamlines,
- (b) curves of constant vorticity,
- (c) curves of constant energy,
- (d) curves of constant pressure,
- (e) curves of constant speed,
- (f) curves of constant flow inclination.

Govindaraju [1972, 1981] used Martin's system that has vorticity, kinetic energy and pressure functions as the dependent variables and studied some of these fifteen confluent flows.

Since there are nine physically important families of curves in MHD, we have thirty-six confluent flows in MHD. Labropulu and Chandna [1990] and Chandna and Labropulu [1990] studied some confluent flows of finitely conducting MHD transverse-aligned and MHD aligned flows respectively, after developing the bar systems for these flows. Labropulu and Chandna [1993] used the bar system developed by Chandna and Kaloni earlier and studied some confluent flows of Cosserat fluids.

1.4 VISCOELASTIC SECOND GRADE FLUID.

In real life, there are many materials that exhibit the mechanical characteristics of both elasticity and viscosity. These materials are called non-Newtonian or viscoelastic. Examples of viscoelastic fluids are polymers, paints, latex, honey, ceramics and liquid metals. These fluids cannot be described satisfactorily by the theory of elasticity or viscosity but by a combination of both. Some of the features that are commonly observed in the viscoelastic fluids are:

- (a) the shear rate dependence; the viscosity decreases with increasing shear rate;
- (b) the normal stress effects, unequal normal stresses in the different directions;
- (c) the high elasticity recovery in shear.

Because of the great diversity in the physical structure of non-Newtonian fluids, it is not possible to describe their mechanical behaviour by a single constitutive equation. For this reason, many constitutive equations have been proposed. The Rivlin-Ericksen model [1955] and Noll's simple fluid model [1958] are among those that have received considerable attention from both experimentalists and theorists. In this work, we adopt the approximate second grade fluid model given by Coleman and Noll [1960]. The constitutive equation of the second grade fluid is given by

$$\underline{\underline{T}} = -p\underline{\underline{I}} + \mu\underline{\underline{A}}_1 + \alpha_1\underline{\underline{A}}_2 + \alpha_2\underline{\underline{A}}_1^2$$

where $\underline{\underline{T}}$ is the stress tensor, $\underline{\underline{I}}$ is the unit tensor, p is the fluid pressure, $\underline{\underline{A}}_1$, $\underline{\underline{A}}_2$ are the Rivlin-Ericksen tensors (defined in chapter II), μ is the shear-dependent viscosity, and α_1 , α_2 are the stress moduli governing the normal stress effects. Markovitz and Coleman [1964] derived methods for measuring these constants and proved that the material constant α_1 is negative. As reported by these authors in the same work, Markovitz and Brown [1962] determined experimentally the values of α_1 and α_2 for the 5.4% solution of a polyisobutylene in cetane at 30°C to be -0.2 g/cm and 1.0 g/cm respectively.

Tanner [1966] and Huilgol [1973] have each considered the steady plane flow in which inertia terms were neglected. Beard and Walters [1964] used the boundary layer equations of a second grade fluid to study stagnation point flow. Fosdick and Rajagopal [1978, 1979] have studied certain anomalous features as well as the uniqueness and the drag in steady motion. Siddiqui and Kaloni [1983], Kaloni and Huschilt [1984] and Kaloni and Siddiqui [1986] employed Martin's approach and inverse methods to obtain some exact solutions of second-grade fluid flow. Rao and Rao [1985] studied the MHD flow of a second-grade fluid in an orthogonal rheometer. Dorrepaal, Chandna and Labropulu [1992] obtained numerical solutions to second-grade fluid flow obliquely impinging on a flat rigid wall. Labropulu, Dorrepaal and Chandna [1993] obtained numerical solutions to second grade fluid flow obliquely impinging on a porous wall with suction or blowing on the wall.

1.5 OUTLINE OF THE PRESENT WORK.

This dissertation contains two parts: analytical solutions of steady plane MHD flows and numerical solutions of viscoelastic second grade fluid flow. The first part can be further subdivided into (i) investigation of exact solutions of various MHD flows using a new approach (Chapters 3 to 6) and (ii) investigation of confluent flows of MHD transverse-aligned fluid flow (Chapter 7). The second part (Chapter 8) is devoted to the numerical solution of second-grade fluid flow obliquely impinging on a porous wall with suction or blowing on the wall.

In the first part of the work, we investigate exact solutions of infinitely conducting and finitely conducting MHD fluid flows. Even though the assumption of infinite conductivity is not believed to be appropriate for most problems in this field (c.f. Resler and Sears [1958]), it should be appropriate in other situations which involve greater 'magnetic Reynolds numbers'. Such situations may include flows involving high gas temperatures, or flows of liquid metals. Our accounting for finite electrical conductivity is motivated by the fact that most fluids have finite electrical conductivity.

In the investigation of exact solutions of MHD flows, we pose and answer the following two questions for steady plane flow:

(I) Given a family of plane curves $l(x,y) = \text{constant}$, can a viscous fluid of constant viscosity flow along these curves?

(II) Given a family of streamlines in a viscous fluid flow of constant viscosity, what is the exact integral of the flow defined by the given streamline pattern when this exact integral is defined to be a set of known functions for the velocity components, the pressure function and the magnetic field components which satisfy the flow equations?

To answer these two questions, we develop a new approach which is a fortuitous

extension of Martin's work [1971]. We initiate the investigation of the first question by assuming that fluid flows along the given family of curves $l(x, y) = \text{constant}$ so that $\psi(x, y) = \text{constant}$ along these curves as well. Therefore, fluid flows along $l(x, y) = \text{constant}$ provided there exists some function $\gamma(\psi)$ such that

$$l(x, y) = \gamma(\psi); \quad \gamma'(\psi) \neq 0$$

where $\gamma'(\psi)$ is the derivative of $\gamma(\psi)$. Fluid flows along $l(x, y) = \text{constant}$ if the solution obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$. The form of $l(x, y)$ in this equation may be such that

(a) for every x we have a unique y on each individual streamline so that $y = y(x, \psi)$ or, for every y we have a unique x on each individual streamline so that $x = x(y, \psi)$

or,

(b) neither for every x we have a unique y nor for every y we have a unique x on each individual streamline.

If case (a) holds true, then we choose the arbitrary family of curves $\phi(x, y) = \text{constant}$ in (ϕ, ψ) -coordinates to be $\phi(x, y) = x$ or $\phi(x, y) = y$ according as $y = y(x, \psi)$ or $x = x(y, \psi)$. The chosen coordinate systems are called von Mises coordinate systems. Successful use of the von Mises coordinates, after deriving flow equations in streamfunction coordinates, has also been made by Barron [1989] and Barron and Naeem [1989] in computational fluid dynamics (CFD). However, if case (a) does not hold true, we proceed in one of the following two ways:

(i) We transform $l(x, y)$ to a different curvilinear coordinate system hoping to be able to proceed as in part (a) with this new coordinate system. We have used polar coordinate system to explain this concept. If the form of $l(x, y) = \gamma(\psi)$ is $l^*(r, \theta) = \gamma(\psi)$ in polar coordinates such that for every r we have a unique θ on each individual streamline, then $\theta = \theta(r, \psi)$ and we choose (r, ψ) -net. On the other

hand, if $l^*(r, \theta) = \gamma(\psi)$ is such that for every θ we have a unique r , then $r = r(\theta, \psi)$ and we choose the (θ, ψ) -net.

(ii) We transform $l(x, y)$ to $L(\xi, \eta)$ where $\xi = \text{Re}[N(z)]$, $\eta = \text{Im}[N(z)]$, $N(z)$ is an analytic function of $z = x + iy$ and use permissible (ξ, ψ) - or (η, ψ) -coordinate system.

In the investigation of confluent flows, we develop the bar system for transverse-aligned MHD fluid flow following Martin's work and study some confluent flows.

In the numerical part of this dissertation, we investigate the effects of elasticity and the effects of suction or blowing on the flow of a second grade fluid which obliquely impinges on a porous wall.

A brief outline of this dissertation is as follows:

Chapter 2 consists of some preliminary work. In section 2.1, we give the flow equations for second grade MHD fluid flow. In section 2.2, we summarize some results of differential geometry needed for our work. In section 2.3, we give a brief outline of Martin's approach [1971]. Section 2.4 is devoted to boundary conditions for the magnetic field.

Chapter 3 is devoted to the investigation of exact solutions of infinitely conducting MHD aligned, finitely conducting MHD aligned and non-MHD flows using Cartesian coordinates. Here $l(x, y)$ is of the form $\frac{y - f(x)}{g(x)}$ and we use the von Mises (x, ψ) -net. In section 3.2, we give the governing equations of infinitely conducting MHD aligned flows, finitely conducting MHD aligned flows and non-MHD flows. We transform the governing equations into the (ϕ, ψ) -net in the next section. In section 3.4, we outline the method and transform the equations into (x, ψ) -net. Section 3.5 is devoted to the applications of the method through various examples. In this section, we also pose and solve some boundary problems using the exact solutions obtained. Finally, the flow patterns for the flows are given.

In Chapter 4, we assume that the function $l(x, y)$ is of the form $\frac{\theta - f(r)}{g(r)}$ in polar coordinates. We use (r, ψ) -coordinates and obtain exact solutions. In section 4.2, we give the flow equations and outline the method. In this section, we also recast the flow equations in the (r, ψ) -net. In section 4.3, we give several examples to implement the method. Finally, the streamline patterns for the flows that we studied are given at the end of this chapter.

Chapter 5 is devoted to the study of exact solutions of infinitely conducting MHD aligned flow using (ξ, ψ) - or (η, ψ) -net where $N(z) = \xi(x, y) + i\eta(x, y)$ is an analytic function of z . In this chapter, the form of $l(x, y)$ is either $\frac{\eta - f(\xi)}{g(\xi)}$ or $\frac{\xi - f(\eta)}{g(\eta)}$. In section 5.2, we give the flow equations of infinitely conducting MHD aligned flow. In section 5.3, we outline the method for both (ξ, ψ) -net and (η, ψ) -net and transform the equations from (ϕ, ψ) -net to (ξ, ψ) -net and (η, ψ) -net. Various applications are given in section 5.4. Streamline patterns for the possible flows are given at the end of the chapter. From some of the examples studied in this chapter, we conclude that Hamel's problem for infinitely conducting MHD aligned flow has at least two more possible solutions in addition to the four known flow patterns for Hamel's problem in ordinary viscous fluid flow.

Chapter 6 is devoted to the study of exact solutions of infinitely conducting MHD orthogonal and variably-inclined flows using the von Mises coordinates (x, ψ) . The governing equations of steady plane infinitely conducting MHD variably-inclined flow are given in section 6.2. In section 6.3, we transform the governing equations into (ϕ, ψ) -net. In section 6.3 we study MHD orthogonal flow. In this section, we try to find all possible functions $f(x)$ and $g(x)$ such that the family of curves $\frac{y - f(x)}{g(x)} = \text{constant}$ are a family of streamlines. We found that there does not exist any infinitely conducting rotational MHD orthogonal flow with streamlines given by $\frac{y - f(x)}{g(x)} = \text{constant}$ and the only possible irrotational flows are the ones with

parallel straight lines or concurrent straight lines as their streamlines. In section 6.4, we study two examples for infinitely conducting MHD variably-inclined flow to show the applicability of our new approach to the general steady plane MHD flow.

In Chapter 7, we study confluent flows of finitely conducting MHD transverse-aligned flow. The equations governing steady plane finitely conducting fluid flow in the presence of a transverse-aligned magnetic field are given in section 7.2. Transformations of these equations to (ϕ, ψ) -net and to the so-called bar system are given in section 7.3. In sections 7.4 and 7.5, we consider the applications of the results of section 7.3. by studying flows in which certain conditions are placed *a priori*.

In Chapter 8, we investigate the flow of a viscoelastic second grade fluid obliquely impinging on a porous wall with suction or blowing. In particular, we want to find out the effects of elasticity on this fluid flow. In section 8.2 we nondimensionalize the governing equations of a second grade fluid and introduce the Weissenberg number which is the ratio of elastic effects to viscous effects. In section 8.3, we use a shooting method and obtain a numerical solution of second grade fluid flow impinging orthogonally on a porous wall with suction or blowing on the wall. Section 8.4 is devoted to the numerical solution of the oblique flow. Discussion of the results for both suction and blowing is given in section 8.5.

Finally, conclusions of all the results obtained and the scope of some future work are given in Chapter 9.

CHAPTER 2

PRELIMINARIES

2.1 BASIC EQUATIONS.

The steady plane flow of a viscous incompressible non-Newtonian second grade electrically conducting fluid, in the presence of a magnetic field, is governed by the following system of equations (c.f. Pai [1962]):

$$\operatorname{div} \mathbf{V} = 0 \quad (\text{continuity}) \quad (2.1)$$

$$\rho(\mathbf{V} \cdot \operatorname{grad}) \mathbf{V} = \operatorname{div} \underline{\underline{T}} + \mu^*(\operatorname{curl} \mathbf{H}) \times \mathbf{H} \quad (\text{linear momentum}) \quad (2.2)$$

$$\operatorname{curl}(\mathbf{V} \times \mathbf{H}) - \frac{1}{\mu^* \sigma} \operatorname{curl}(\operatorname{curl} \mathbf{H}) = 0 \quad (\text{diffusion}) \quad (2.3)$$

where the constitutive equation for the Cauchy stress $\underline{\underline{T}}$ is given by Coleman and Noll [1960]

$$\underline{\underline{T}} = -p \underline{\underline{I}} + \mu \underline{\underline{A}}_1 + \alpha_1 \underline{\underline{A}}_2 + \alpha_2 \underline{\underline{A}}_1^2 \quad (2.4)$$

The Rivlin-Ericksen tensors $\underline{\underline{A}}_1$ and $\underline{\underline{A}}_2$ are defined as

$$\underline{\underline{A}}_1 = (\operatorname{grad} \mathbf{V}) + (\operatorname{grad} \mathbf{V})^T \quad (2.5)$$

$$\underline{\underline{A}}_2 = (\mathbf{V} \cdot \operatorname{grad}) \underline{\underline{A}}_1 + (\operatorname{grad} \mathbf{V})^T \underline{\underline{A}}_1 + \underline{\underline{A}}_1 (\operatorname{grad} \mathbf{V}) \quad (2.6)$$

Equations (2.1) to (2.3), with $\underline{\underline{T}}$ given by (2.4) to (2.6), is a system of seven equations where \mathbf{V} denotes the velocity vector field, \mathbf{H} the magnetic vector field, p the fluid

pressure function, ρ the constant fluid density, μ the constant coefficient of viscosity, μ^* the constant magnetic permeability, σ the constant electrical conductivity, I the unit tensor and α_1, α_2 the normal-stress moduli.

Taking $\mathbf{V} = (u, v, w)$ in Cartesian coordinates (x, y, z) , we have

$$\text{grad}\mathbf{V} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix}$$

and

$$\begin{aligned} \text{grad}A_1 &= \frac{\partial A_1}{\partial x} \mathbf{i} + \frac{\partial A_1}{\partial y} \mathbf{j} + \frac{\partial A_1}{\partial z} \mathbf{k} \\ &= \begin{pmatrix} 2\frac{\partial^2 u}{\partial x^2} & \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} \\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} & 2\frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 w}{\partial x^2} & \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 w}{\partial x \partial y} & 2\frac{\partial^2 w}{\partial x \partial z} \end{pmatrix} \mathbf{i} \\ &+ \begin{pmatrix} 2\frac{\partial^2 u}{\partial x \partial y} & \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x \partial y} & \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y} \\ \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} & 2\frac{\partial^2 v}{\partial y^2} & \frac{\partial^2 v}{\partial z \partial y} + \frac{\partial^2 w}{\partial y^2} \\ \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 w}{\partial x \partial y} & \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2} & 2\frac{\partial^2 w}{\partial z \partial y} \end{pmatrix} \mathbf{j} \\ &+ \begin{pmatrix} 2\frac{\partial^2 u}{\partial x \partial z} & \frac{\partial^2 u}{\partial y \partial z} + \frac{\partial^2 v}{\partial x \partial z} & \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} \\ \frac{\partial^2 v}{\partial x \partial z} + \frac{\partial^2 u}{\partial y \partial z} & 2\frac{\partial^2 v}{\partial y \partial z} & \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z} \\ \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 w}{\partial x \partial z} & \frac{\partial^2 v}{\partial z^2} + \frac{\partial^2 w}{\partial y \partial z} & 2\frac{\partial^2 w}{\partial z^2} \end{pmatrix} \mathbf{k} \end{aligned}$$

where \mathbf{i}, \mathbf{j} and \mathbf{k} are the unit vectors in Cartesian coordinates.

The magnetic field \mathbf{H} must satisfy an additional equation

$$\text{div} \mathbf{H} = 0 \quad (\text{solenoidal}) \quad (2.7)$$

which expresses the absence of magnetic poles in the flow.

When $\alpha_1 = \alpha_2 = 0$, equation (2.4) is the Cauchy stress for a Newtonian fluid. Furthermore, when $\mathbf{H} = 0$, equations (2.3), (2.7) are identically satisfied and equations (2.1), (2.2), with \mathbf{T} given by (2.4), describe the motion of a non-Newtonian, non-MHD fluid.

2.1.1 Plane Motion of MHD Newtonian Fluid Flow.

We consider steady plane flow of an electrically conducting fluid having finite electrical conductivity in the presence of a magnetic field. Letting $\alpha_1 = \alpha_2 = 0$, $\mathbf{V} = (u(x, y), v(x, y), 0)$, $\mathbf{H} = (H_1(x, y), H_2(x, y), H_3(x, y))$, and making use of (2.4) to (2.6), equations (2.1) to (2.3) and (2.7) take the form (c.f. Pai [1962]):

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} &= \mu \nabla^2 u - \mu^* \left[H_2 \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + H_3 \frac{\partial H_3}{\partial x} \right] \\ \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} &= \mu \nabla^2 v + \mu^* \left[H_1 \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) - H_3 \frac{\partial H_3}{\partial y} \right] \\ H_2 \frac{\partial H_3}{\partial y} + H_1 \frac{\partial H_3}{\partial x} &= 0 \\ u H_2 - v H_1 &= \frac{1}{\mu^* \sigma} \left(\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \right) + C \\ u \frac{\partial H_3}{\partial x} + v \frac{\partial H_3}{\partial y} - \frac{1}{\mu^* \sigma} \left(\frac{\partial^2 H_3}{\partial x^2} + \frac{\partial^2 H_3}{\partial y^2} \right) &= 0 \\ \frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} &= 0 \end{aligned} \quad (2.8)$$

This is a system of seven equations in six unknowns u , v , H_1 , H_2 , H_3 and p as functions of x , y . The constant C is an arbitrary constant of integration obtained

by integrating the x- and y-components of the vector diffusion equation (2.3). This constant C is equal to zero only if the fluid is infinitely conducting and the plane projection of the spatial magnetic field on the plane of flow is aligned.

Introducing ω and Ω given by

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}, \quad \Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad (2.9)$$

system (2.8) can be replaced by the following system of equations(c.f. Labropoulou and Chandna [1989]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.10)$$

$$\frac{\partial}{\partial x} \left(\frac{1}{2} \rho q^2 + p + \frac{1}{2} \mu^* H_3^2 \right) + \mu \frac{\partial \omega}{\partial y} - \rho v \omega + \mu^* H_2 \Omega = 0 \quad (2.11)$$

$$\frac{\partial}{\partial y} \left(\frac{1}{2} \rho q^2 + p + \frac{1}{2} \mu^* H_3^2 \right) - \mu \frac{\partial \omega}{\partial x} + \rho u \omega - \mu^* H_1 \Omega = 0 \quad (2.12)$$

$$H_2 \frac{\partial H_3}{\partial y} + H_1 \frac{\partial H_3}{\partial x} = 0 \quad (2.13)$$

$$u H_2 - v H_1 = \frac{1}{\mu^* \sigma} \Omega + C \quad (2.14)$$

$$u \frac{\partial H_3}{\partial x} + v \frac{\partial H_3}{\partial y} - \frac{1}{\mu^* \sigma} \left(\frac{\partial^2 H_3}{\partial x^2} + \frac{\partial^2 H_3}{\partial y^2} \right) = 0 \quad (2.15)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \quad (2.16)$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (2.17)$$

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad (2.18)$$

These nine equations give u , v , H_1 , H_2 , H_3 , p , ω and Ω as functions of x , y . Although the number of equations and unknowns have increased by two, the order of the linear momentum equations has decreased from two to one. Here $q = \sqrt{u^2 + v^2}$ is the speed.

2.1.2 Plane Motion of non-Newtonian Fluid Flow.

We consider steady plane flow of a non-Newtonian second-grade fluid. Letting $\mathbf{V} = (u(x, y), v(x, y), 0)$, $\mathbf{H} = 0$ and making use of (2.4) to (2.6), equations (2.1) and (2.2) take the form (Kalonji and Siddiqui [1983]):

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2.19)$$

$$\begin{aligned} \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} = & \mu \nabla^2 u + \alpha_1 \left\{ \frac{\partial}{\partial x} \left[2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left(\frac{\partial u}{\partial x} \right)^2 \right. \right. \\ & + 2 \frac{\partial v}{\partial x} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left. \right] + \frac{\partial}{\partial y} \left[\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right. \\ & + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \left. \right] \left. \right\} + \alpha_2 \frac{\partial}{\partial x} \left[4 \left(\frac{\partial u}{\partial x} \right)^2 \right. \\ & \left. + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \end{aligned} \quad (2.20)$$

$$\begin{aligned} \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} = & \mu \nabla^2 v + \alpha_1 \left\{ \frac{\partial}{\partial x} \left[\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right. \right. \\ & + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \left. \right] + \frac{\partial}{\partial y} \left[2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial x^2} + 4 \left(\frac{\partial v}{\partial y} \right)^2 \right. \\ & + 2 \frac{\partial u}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left. \right] \left. \right\} + \alpha_2 \frac{\partial}{\partial y} \left[4 \left(\frac{\partial v}{\partial y} \right)^2 \right. \\ & \left. + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \end{aligned} \quad (2.21)$$

This is a system of three equations in three unknowns u , v and p as functions of x , y .

2.2 SOME RESULTS FROM DIFFERENTIAL GEOMETRY.

Let

$$x = x(\phi, \psi), \quad y = y(\phi, \psi) \quad (2.22)$$

define a system of curvilinear coordinates (ϕ, ψ) in the (x, y) -plane with the squared element of arc length along any curve given by

$$ds^2 = E(\phi, \psi) d\phi^2 + 2F(\phi, \psi) d\phi d\psi + G(\phi, \psi) d\psi^2 \quad (2.23)$$

where

$$E = \left(\frac{\partial x}{\partial \phi} \right)^2 + \left(\frac{\partial y}{\partial \phi} \right)^2, \quad F = \frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \psi}, \quad G = \left(\frac{\partial x}{\partial \psi} \right)^2 + \left(\frac{\partial y}{\partial \psi} \right)^2 \quad (2.24)$$

Equations (2.22) can be solved to obtain $\phi = \phi(x, y)$, $\psi = \psi(x, y)$ such that

$$\frac{\partial x}{\partial \phi} = J \frac{\partial \psi}{\partial y}, \quad \frac{\partial x}{\partial \psi} = -J \frac{\partial \phi}{\partial y}, \quad \frac{\partial y}{\partial \phi} = -J \frac{\partial \psi}{\partial x}, \quad \frac{\partial y}{\partial \psi} = J \frac{\partial \phi}{\partial x} \quad (2.25)$$

provided $0 < |J| < \infty$, where J is the transformation Jacobian, and

$$J = \frac{\partial x}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \phi} = \pm \sqrt{EG - F^2} = \pm W \text{ (say)} \quad (2.26)$$

Let i) $P(x, y)$ be any point on the curve $\psi = c_1$, where c_1 is a constant (see Figure 2.1),

ii) variable ϕ be increasing on $\psi = c_1$ in the direction in which x, y are increasing.

iii) $\alpha(x, y)$ or $\alpha(\phi, \psi)$ denote the angle of inclination of the tangent to the coordinate line $\psi = c_1$, directed in the sense of increasing ϕ .

The tangent vector to $\psi = c_1$ at P is $\left(\frac{\partial x}{\partial \phi}, \frac{\partial y}{\partial \phi} \right)$, and we have

$$\tan \alpha = \frac{\frac{\partial y}{\partial \phi}}{\frac{\partial x}{\partial \phi}} \quad \text{or} \quad \frac{\partial x}{\partial \phi} \sin \alpha = \frac{\partial y}{\partial \phi} \cos \alpha \quad (2.27)$$

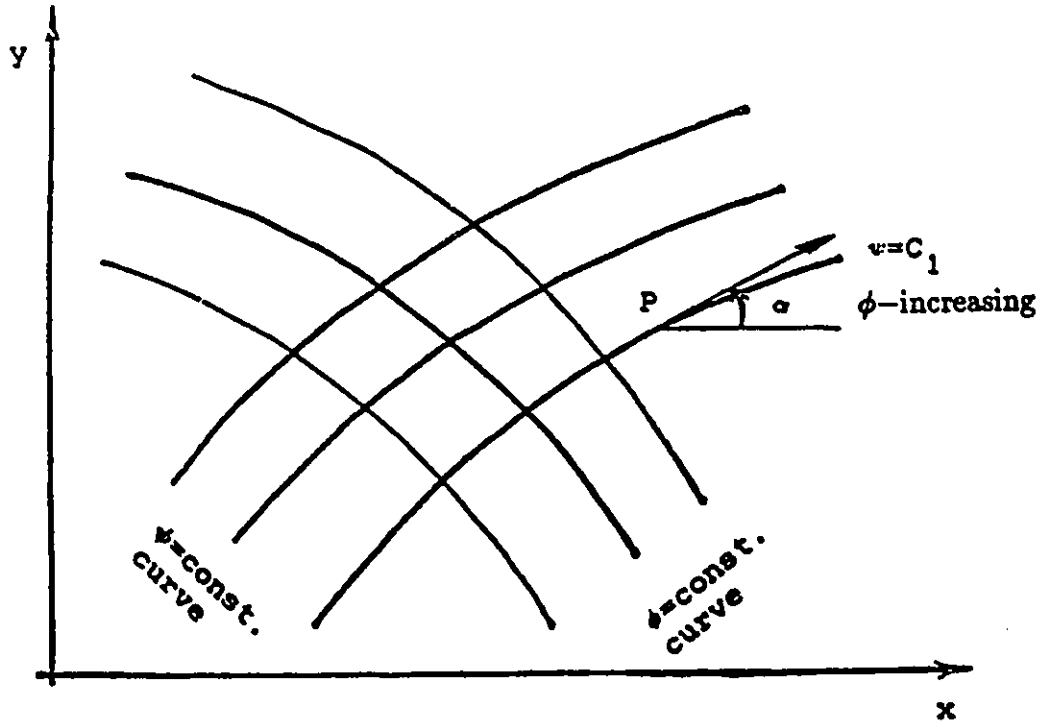


Figure 2.1: The (ϕ, ψ) -curvilinear net

Using (2.27) in the first equation of (2.24), we get

$$\frac{\partial x}{\partial \phi} = \sqrt{E} \cos \alpha, \quad \frac{\partial y}{\partial \phi} = \sqrt{E} \sin \alpha \quad (2.28)$$

The first two equations in (2.24) can be rewritten in the form

$$\frac{\partial x}{\partial \phi} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \phi} \frac{\partial y}{\partial \phi} = E$$

$$\frac{\partial x}{\partial \psi} \frac{\partial x}{\partial \phi} + \frac{\partial y}{\partial \psi} \frac{\partial y}{\partial \phi} = F$$

Solving these two equations for $\frac{\partial x}{\partial \phi}$, we obtain

$$J \frac{\partial x}{\partial \phi} = E \frac{\partial y}{\partial \psi} - F \frac{\partial y}{\partial \phi}$$

or

$$E \frac{\partial y}{\partial \psi} = J \frac{\partial x}{\partial \phi} + F \frac{\partial y}{\partial \phi} \quad (2.29)$$

Similarly, we find that

$$E \frac{\partial x}{\partial \psi} = F \frac{\partial x}{\partial \phi} - J \frac{\partial y}{\partial \phi} \quad (2.30)$$

Using (2.28) in (2.29) and (2.30), we get

$$\begin{aligned} \frac{\partial x}{\partial \psi} &= \frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \\ \frac{\partial y}{\partial \psi} &= \frac{J}{\sqrt{E}} \cos \alpha + \frac{F}{\sqrt{E}} \sin \alpha \end{aligned} \quad (2.31)$$

Differentiating (2.26) with respect to ϕ , we have

$$\frac{\partial J}{\partial \phi} = \frac{G \frac{\partial E}{\partial \phi} + E \frac{\partial G}{\partial \phi} - 2F \frac{\partial F}{\partial \phi}}{2J} \quad (2.32)$$

The integrability conditions $\frac{\partial^2 x}{\partial \phi \partial \psi} = \frac{\partial^2 x}{\partial \psi \partial \phi}$, $\frac{\partial^2 y}{\partial \phi \partial \psi} = \frac{\partial^2 y}{\partial \psi \partial \phi}$ give

$$\begin{aligned} -\sqrt{E} \sin \alpha \frac{\partial \alpha}{\partial \psi} + \left[\frac{F}{\sqrt{E}} \sin \alpha + \frac{J}{\sqrt{E}} \cos \alpha \right] \frac{\partial \alpha}{\partial \phi} &= \left[-\frac{1}{2\sqrt{E}} \frac{\partial E}{\partial \psi} - \frac{F}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} \right. \\ &\quad \left. + \frac{1}{\sqrt{E}} \frac{\partial F}{\partial \phi} \right] \cos \alpha + \left[\frac{J}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}} \frac{\partial J}{\partial \phi} \right] \sin \alpha \end{aligned}$$

and

$$\begin{aligned} \sqrt{E} \cos \alpha \frac{\partial \alpha}{\partial \psi} - \left[\frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \right] \frac{\partial \alpha}{\partial \phi} &= \left[-\frac{1}{2\sqrt{E}} \frac{\partial E}{\partial \psi} - \frac{F}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} \right. \\ &\quad \left. + \frac{1}{\sqrt{E}} \frac{\partial F}{\partial \phi} \right] \sin \alpha - \left[\frac{J}{2E\sqrt{E}} \frac{\partial E}{\partial \phi} - \frac{1}{\sqrt{E}} \frac{\partial J}{\partial \phi} \right] \cos \alpha \end{aligned}$$

Solving the above two equations for $\frac{\partial \alpha}{\partial \phi}$, $\frac{\partial \alpha}{\partial \psi}$ and using equations (2.32), we obtain

$$\frac{\partial \alpha}{\partial \phi} = \frac{1}{2EJ} \left[-F \frac{\partial E}{\partial \phi} + 2E \frac{\partial F}{\partial \phi} - E \frac{\partial E}{\partial \psi} \right]$$

$$\frac{\partial \alpha}{\partial \psi} = \frac{1}{2EJ} \left[-F \frac{\partial E}{\partial \psi} + E \frac{\partial G}{\partial \phi} \right]$$

which can be written as

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma_{11}^2, \quad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E} \Gamma_{12}^2 \quad (2.33)$$

where

$$\begin{aligned}\Gamma_{11}^2 &= \frac{1}{2W^2} \left[-F \frac{\partial E}{\partial \phi} + 2E \frac{\partial F}{\partial \phi} - E \frac{\partial E}{\partial \psi} \right] \\ \Gamma_{12}^2 &= \frac{1}{2W^2} \left[E \frac{\partial G}{\partial \phi} - F \frac{\partial E}{\partial \psi} \right]\end{aligned}\tag{2.34}$$

From (2.33), we see that the integrability condition $\frac{\partial^2 \alpha}{\partial \phi \partial \psi} = \frac{\partial^2 \alpha}{\partial \psi \partial \phi}$ implies that

$$\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) = 0\tag{2.35}$$

Equation (2.35) says that the Gaussian curvature

$$K = \frac{1}{W} \left[\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) \right] = 0$$

and is referred to as the Gauss equation.

Conversely, if E, F, G are given as functions of ϕ, ψ such that the Gauss equation (2.35) is satisfied, then we show that the functions $x(\phi, \psi)$ and $y(\phi, \psi)$ can be obtained in terms of E, F and G where E, F, G satisfy (2.23).

Equation (2.35) implies the existence of $\alpha = \alpha(\phi, \psi)$ such that

$$\frac{\partial \alpha}{\partial \phi} = \frac{J}{E} \Gamma_{11}^2, \quad \frac{\partial \alpha}{\partial \psi} = \frac{J}{E} \Gamma_{12}^2$$

Therefore, α can be obtained from

$$\alpha = \int \left(\frac{\partial \alpha}{\partial \phi} d\phi + \frac{\partial \alpha}{\partial \psi} d\psi \right) = \int \frac{J}{E} (\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi)\tag{2.36}$$

The functions $x(\phi, \psi)$ and $y(\phi, \psi)$ are given by

$$\begin{aligned}x &= \int \left\{ \left(\sqrt{E} \cos \alpha \right) d\phi + \left(\frac{F}{\sqrt{E}} \cos \alpha - \frac{J}{\sqrt{E}} \sin \alpha \right) d\psi \right\} \\ y &= \int \left\{ \left(\sqrt{E} \sin \alpha \right) d\phi + \left(\frac{F}{\sqrt{E}} \sin \alpha + \frac{J}{\sqrt{E}} \cos \alpha \right) d\psi \right\}\end{aligned}\tag{2.37}$$

Introducing the complex variable $z = x + iy$, equations (2.37) can be written in a concise form as

$$z = \int \frac{1}{\sqrt{E}} e^{i\alpha} \{ E d\phi + (F + iJ) d\psi \}\tag{2.38}$$

where α is given by (2.36).

Summing up, we have:

Theorem 2.1. Three functions E, F, G of ϕ, ψ serve as coefficients in the first fundamental form

$$ds^2 = E d\phi^2 + 2F d\phi d\psi + G d\psi^2$$

for a plane with a curvilinear coordinate system

$$x = x(\phi, \psi), \quad y = y(\phi, \psi)$$

if and only if they satisfy the Gauss equation

$$\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) = 0$$

If this condition is satisfied, then the functions $x(\phi, \psi)$ and $y(\phi, \psi)$ that define the curvilinear coordinate system, are given in terms of E, F, G by (2.38).

From the relation

$$W = \sqrt{EG - F^2}$$

we find that

$$\begin{aligned} \frac{\partial}{\partial \phi} \left(\frac{E}{2W^2} \right) &= \frac{1}{2W^2} \left\{ \frac{\partial E}{\partial \phi} - \frac{E}{W^2} \left[E \frac{\partial G}{\partial \phi} + G \frac{\partial E}{\partial \phi} - 2F \frac{\partial F}{\partial \phi} \right] \right\} \\ &= \frac{1}{W^2} [F \Gamma_{11}^2 - E \Gamma_{12}^2] \end{aligned} \quad (2.39)$$

$$\frac{\partial}{\partial \psi} \left(\frac{E}{2W^2} \right) = \frac{1}{W^2} [F \Gamma_{12}^2 - E \Gamma_{22}^2] \quad (2.40)$$

$$\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) = \frac{1}{W} (G \Gamma_{11}^2 - 2F \Gamma_{12}^2 + E \Gamma_{22}^2) \quad (2.41)$$

where Γ_{22}^2 is given by

$$\Gamma_{22}^2 = \frac{1}{2W^2} \left[E \frac{\partial G}{\partial \psi} - 2F \frac{\partial F}{\partial \psi} + F \frac{\partial G}{\partial \phi} \right] \quad (2.42)$$

Corollary 2.1. *If $\phi(x,y)=\text{constant}$ and $\psi(x,y)=\text{constant}$ form an orthogonal curvilinear net, then $F = 0$ and the two functions $E(\phi, \psi)$, $G(\phi, \psi)$ serve as coefficients in the first fundamental form*

$$ds^2 = E d\phi^2 + G d\psi^2$$

for a plane with a curvilinear coordinate system

$$x = x(\phi, \psi), \quad y = y(\phi, \psi)$$

if and only if they satisfy the Gauss equation

$$\frac{\partial}{\partial \psi} \left(\frac{1}{J} \frac{\partial E}{\partial \psi} \right) - \frac{\partial}{\partial \phi} \left(\frac{1}{J} \frac{\partial G}{\partial \phi} \right) = 0$$

If this condition is satisfied, then the functions $x(\phi, \psi)$ and $y(\phi, \psi)$, are given in terms of E , G by

$$z = \int \frac{1}{\sqrt{E}} e^{i\alpha} \{ E d\phi + iJ d\psi \}$$

where

$$\alpha = \frac{1}{2} \int \left\{ - \left(\frac{1}{J} \frac{\partial E}{\partial \psi} \right) d\phi + \left(\frac{1}{J} \frac{\partial G}{\partial \phi} \right) d\psi \right\}$$

2.3 MARTIN'S APPROACH.

Taking $\alpha_1 = \alpha_2 = 0$ in equations (2.8) to (2.10), we find that the steady plane viscous incompressible Newtonian fluid flow is governed by (c.f. Yih [1979]) :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{continuity}) \quad (2.43)$$

$$\left. \begin{aligned} \rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) + \frac{\partial p}{\partial x} &= \mu \nabla^2 u & (\text{linear} \\ \rho \left(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) + \frac{\partial p}{\partial y} &= \mu \nabla^2 v & (\text{momentum}) \end{aligned} \right\} \quad (2.44)$$

of three equations for u , v and p as functions of x , y .

A first order system that may replace this second order system (2.43) to (2.44) is [c.f. Martin, 1971]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{continuity}) \quad (2.45)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \quad (\text{vorticity}) \quad (2.46)$$

$$\left. \begin{aligned} \mu \frac{\partial \omega}{\partial y} - \rho v \omega &= -\frac{\partial h}{\partial x} & (\text{linear} \\ \mu \frac{\partial \omega}{\partial x} - \rho u \omega &= \frac{\partial h}{\partial y} & (\text{momentum}) \end{aligned} \right\} \quad (2.47)$$

$$h = \frac{1}{2} \rho (u^2 + v^2) + p \quad (\text{energy}) \quad (2.48)$$

for the five unknown functions u , v , ω , h and p as functions of x , y .

Equation of continuity implies the existence of a streamfunction $\psi = \psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \quad (2.49)$$

We take two families of curves $\phi(x, y) = \text{constant}$ and $\psi(x, y) = \text{constant}$ such that a curvilinear net is defined in the physical plane when $\psi = \psi(x, y)$ is the streamfunction and $\phi = \phi(x, y)$ is an arbitrary function.

Having recorded the results from differential geometry in section 2.2, we transform equations (2.45) to (2.48) into new form with new variables ϕ , ψ .

Continuity Equation: From equations (2.49) and (2.25), we have

$$\frac{\partial x}{\partial \phi} = Ju, \quad \frac{\partial y}{\partial \phi} = Jv \quad (2.50)$$

Equations (2.50) in (ϕ, ψ) coordinates are equivalent to the continuity equation (2.45) in (x, y) coordinates.

If we introduce polar coordinates q, θ in the hodograph plane by placing

$$u = q \cos \theta, \quad v = q \sin \theta$$

where θ is the direction of the flow in the physical plane, equation (2.50) becomes

$$\frac{\partial x}{\partial \phi} = qJ \cos \theta, \quad \frac{\partial y}{\partial \phi} = qJ \sin \theta \quad (2.51)$$

When equations (2.51) are compared with (2.28) two alternatives arise, namely

$$\text{i) } \theta = \alpha, \quad qJ = \sqrt{E}, \quad J > 0$$

$$\text{ii) } \theta = \alpha + \pi, \quad qJ = -\sqrt{E}, \quad J < 0$$

In i) the fluid flows towards higher parameter values of ϕ and in ii) the fluid flows towards lower parameter values of ϕ . In either case, from (2.26) we have

$$qW = \sqrt{E}$$

This relation is, therefore, a consequence of the continuity equation since when it holds true, equations (2.28) yield

$$\frac{\partial x}{\partial \phi} = qW \cos \alpha, \quad \frac{\partial y}{\partial \phi} = qW \sin \alpha$$

and these equations imply equations (2.51) which are equivalent to the continuity equation.

Vorticity Equation: Transforming equation (2.46) in the (ϕ, ψ) -net, we have

$$\omega = \frac{\partial v}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial v}{\partial \psi} \frac{\partial \psi}{\partial x} - \frac{\partial u}{\partial \phi} \frac{\partial \phi}{\partial y} - \frac{\partial u}{\partial \psi} \frac{\partial \psi}{\partial y} \quad (2.52)$$

Employing (2.25) in (2.52), we get

$$\omega = \frac{1}{J} \left\{ \frac{\partial y}{\partial \psi} \frac{\partial v}{\partial \phi} - \frac{\partial y}{\partial \phi} \frac{\partial v}{\partial \psi} + \frac{\partial x}{\partial \psi} \frac{\partial u}{\partial \phi} - \frac{\partial x}{\partial \phi} \frac{\partial u}{\partial \psi} \right\} \quad (2.53)$$

Using (2.50) in (2.53), we obtain

$$\omega = \frac{1}{J} \left\{ \frac{\partial y}{\partial \psi} \frac{\partial}{\partial \phi} \left(\frac{1}{J} \frac{\partial y}{\partial \phi} \right) - \frac{\partial y}{\partial \phi} \frac{\partial}{\partial \psi} \left(\frac{1}{J} \frac{\partial y}{\partial \phi} \right) + \frac{\partial x}{\partial \psi} \frac{\partial}{\partial \phi} \left(\frac{1}{J} \frac{\partial x}{\partial \phi} \right) - \frac{\partial x}{\partial \phi} \frac{\partial}{\partial \psi} \left(\frac{1}{J} \frac{\partial x}{\partial \phi} \right) \right\} \quad (2.54)$$

Rewriting equation (2.54) and using (2.24), we have

$$\omega = \frac{1}{J} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \right) \right]$$

Proceeding as above we can also transform the linear momentum equations (2.47) in the (ϕ, ψ) -net and the first order system (2.45) to (2.48) of five equations for the five unknown functions $u(x, y)$, $v(x, y)$, $h(x, y)$, $p(x, y)$, $\omega(x, y)$ may be replaced by the system [c.f. Martin, 1971]:

$$q = \frac{\sqrt{E}}{W} \quad (\text{continuity}) \quad (2.55)$$

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \quad (\text{vorticity}) \quad (2.56)$$

$$\mu \left[F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right] = J \frac{\partial h}{\partial \phi} \quad (\text{linear}) \quad (2.57)$$

$$\mu \left[G \frac{\partial \omega}{\partial \phi} - F \frac{\partial \omega}{\partial \psi} \right] = J \frac{\partial h}{\partial \psi} + J \rho \omega \quad (\text{momentum})$$

$$h = \frac{\rho E}{2W^2} + p \quad (\text{energy}) \quad (2.58)$$

$$\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) = 0 \quad (\text{Gauss}) \quad (2.59)$$

of six equations for seven unknown functions E , F , G , q , ω , h and p of ϕ , ψ . Here $J = \pm W$ and the positive or negative sign is taken accordingly as the parameter ϕ increases or decreases along the streamlines in the direction of flow. The system

(2.55) to (2.59) is underdetermined due to the arbitrariness inherent in the choice of the coordinate lines $\phi = \text{constant}$.

We use the integrability condition $\frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi}$ to eliminate h from the linear momentum equations (2.57) and note that the four functions $E(\phi, \psi)$, $F(\phi, \psi)$, $G(\phi, \psi)$ and $\omega(\phi, \psi)$ satisfy the system

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \quad (2.60)$$

$$\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) = 0 \quad (2.61)$$

$$\mu \left[\frac{\partial}{\partial \phi} \left(\frac{G}{W} \frac{\partial \omega}{\partial \phi} - \frac{F}{W} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{W} \frac{\partial \omega}{\partial \psi} - \frac{F}{W} \frac{\partial \omega}{\partial \phi} \right) \right] \mp \rho \frac{\partial \omega}{\partial \phi} = 0 \quad (2.62)$$

of three equations when the negative or positive sign is taken in the last equation accordingly as the parameter ϕ increases or decreases in the direction of flow.

Once a solution

$$E = E(\phi, \psi), \quad F = F(\phi, \psi), \quad G = G(\phi, \psi), \quad \omega = \omega(\phi, \psi)$$

of (2.60) to (2.62) has been obtained, continuity equation (2.55) determines $q = q(\phi, \psi)$, linear momentum equations (2.57) are used to determine $h = h(\phi, \psi)$ from $h = \int \left(\frac{\partial h}{\partial \phi} d\phi + \frac{\partial h}{\partial \psi} d\psi \right)$ and the energy equation (2.58) yields the pressure function $p = h - \frac{1}{2} \rho \frac{E}{W^2}$. Furthermore, the (ϕ, ψ) -plane is mapped on the physical and hodograph planes by

$$z = \int \frac{1}{\sqrt{E}} e^{i\alpha} \{ E d\phi + (F + iJ) d\psi \}, \quad \alpha = \int \frac{J}{E} (\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi)$$

and

$$u + iv = \frac{\sqrt{E}}{J} e^{i\alpha}$$

to complete the solution.

2.4 BOUNDARY CONDITIONS FOR THE MAGNETIC FIELD.

Given a surface S , the boundary conditions for the magnetic field \mathbf{H} are given by (c.f. Dragos [1975]):

$$[\mathbf{H}] \cdot \mathbf{t} = 0 \quad (2.63)$$

$$[\mathbf{H}] \cdot \mathbf{n} = 0 \quad (2.64)$$

where \mathbf{t} and \mathbf{n} are the tangent and the normal vectors to the surface S respectively. Equation (2.63) expresses the continuity condition of the tangential component of the magnetic field and equation (2.64) expresses the continuity condition of the normal component of the magnetic field. In other words, equation (2.63) implies that the tangential component of the magnetic field outside S is equal to the tangential component of the magnetic field inside S and equation (2.64) implies that the normal component of the magnetic field outside S is equal to the normal component of the magnetic field inside S . Thus, we have

$$\mathbf{H}_{\text{outside}} \cdot \mathbf{t} \Big|_S = \mathbf{H}_{\text{inside}} \cdot \mathbf{t} \Big|_S \quad (2.65)$$

$$\mathbf{H}_{\text{outside}} \cdot \mathbf{n} \Big|_S = \mathbf{H}_{\text{inside}} \cdot \mathbf{n} \Big|_S \quad (2.66)$$

CHAPTER 3

EXACT SOLUTIONS OF STEADY PLANE FLOWS USING VON MISES COORDINATES

3.1 INTRODUCTION.

This chapter deals with a new approach for the investigation of exact solutions of i) steady plane viscous infinitely conducting MHD aligned fluid flow, ii) steady plane viscous finitely conducting MHD aligned fluid flow and iii) steady plane viscous non-MHD fluid flow. MHD plane flow is said to be *aligned* if the magnetic field is everywhere parallel to the velocity field. MHD aligned and electromagnetohydrodynamic (EMHD) aligned flows have been extensively investigated in the literature by Sears and Resler [1959], Lary [1962], Chandna and Nath [1972], Chandna, Toews and Prabakaran [1981], Yin [1984], Nguyen and Chandna [1990], Chandna and Labropulu [1990] and Chandna, Husain and Labropulu [1991], to mention a few.

A solution of a flow problem under consideration is said to be *exact* if either this solution is valid for all x and all values of viscosity ν and magnetic permeability μ^* or this solution is valid for all x , for all values of ν and μ^* and it satisfies the boundary conditions of a realistically imposed physical problem. For type (iii) problems, the magnetic permeability is equal to zero.

We pose and answer the following two questions:

- (a) Given a family of plane curves $\frac{y - f(x)}{g(x)} = \text{constant}$, can a fluid of constant viscosity flow along these curves?
- (b) Given a family of streamlines in a fluid flow of constant viscosity, what is the exact integral or exact solution of the flow defined by the given streamline pattern?

We initiate the investigation of our first question by assuming that fluid flows along the given family of curves $\frac{y - f(x)}{g(x)} = \text{constant}$ so that $\psi(x, y) = \text{constant}$ along these curves as well. Therefore, fluid flows along $\frac{y - f(x)}{g(x)} = \text{constant}$ provided there exists some function $\gamma(\psi)$ such that

$$\frac{y - f(x)}{g(x)} = \gamma(\psi); \quad \gamma'(\psi) \neq 0$$

where $\gamma'(\psi)$ is the derivative of $\gamma(\psi)$. Since

$$u = \frac{1}{g(x)\gamma'(\psi)}, \quad v = -\frac{1}{\gamma'(\psi)} \frac{\partial}{\partial x} \left[\frac{y - f(x)}{g(x)} \right]$$

we have to assume that $\gamma'(\psi) \neq 0$.

We choose the von Mises coordinates and, therefore, the arbitrary family of curves $\phi(x, y) = \text{constant}$ is taken to be $\phi(x, y) = x = \text{constant}$. Fluid flows along $\frac{y - f(x)}{g(x)} = \text{constant}$ only if the solution obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$. Having obtained the answer to our first question in the affirmative, we employ $y - f(x) = g(x)\gamma(\psi)$ in the governing equations to obtain the exact integral for the permissible flow pattern.

3.2 FLOW EQUATIONS.

The steady plane flow of a viscous incompressible and electrically conducting fluid, in the presence of a magnetic field, is governed by equations (2.10) to (2.18).

Considering the flow to be aligned, we take

$$H_1 = \beta u, \quad H_2 = \beta v, \quad H_3 \equiv 0 \quad (3.1)$$

where $\beta(x, y)$ is some unknown scalar function.

Introducing

$$h = \frac{1}{2} \rho (u^2 + v^2) + p \quad (3.2)$$

in the two linear momentum equations (2.11) and (2.12), we obtain

$$\frac{\partial h}{\partial x} + \mu \frac{\partial \omega}{\partial y} - \rho v \omega + \mu^* H_2 \Omega = 0$$

$$\frac{\partial h}{\partial y} - \mu \frac{\partial \omega}{\partial x} + \rho u \omega - \mu^* H_1 \Omega = 0$$

In this chapter, we deal with both infinitely and finitely conducting fluids. Diffusion equation (2.14) is identically satisfied for an infinitely conducting MHD aligned flow. However, this equation requires Ω to be a constant, say Ω_0 , for a finitely conducting MHD aligned flow. Thus, we have:

3.2.1 Infinitely Conducting Flow.

An infinitely conducting steady plane MHD aligned flow is governed by the following system of six partial differential equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (3.3)$$

$$\left. \begin{aligned} \frac{\partial h}{\partial x} + \mu \frac{\partial \omega}{\partial y} - \rho v \omega + \mu^* \beta v \Omega &= 0 \\ \frac{\partial h}{\partial y} - \mu \frac{\partial \omega}{\partial x} + \rho u \omega - \mu^* \beta u \Omega &= 0 \end{aligned} \right\} \quad (3.4)$$

$$u \frac{\partial \beta}{\partial x} + v \frac{\partial \beta}{\partial y} = 0 \quad (3.5)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \omega \quad (3.6)$$

$$\beta \omega + v \frac{\partial \beta}{\partial x} - u \frac{\partial \beta}{\partial y} = \Omega \quad (3.7)$$

for six unknown functions $u(x, y)$, $v(x, y)$, $h(x, y)$, $\omega(x, y)$, $\Omega(x, y)$ and $\beta(x, y)$. Once a solution of this system is determined, the pressure function $p(x, y)$ and the magnetic vector field \mathbf{H} are found by using (3.2) and (3.1).

3.2.2 Finitely Conducting Flow.

A finitely conducting steady plane MHD aligned flow is governed by the system (3.3) to (3.7) of six partial differential equations when current density function Ω is replaced by constant Ω_0 in this system.

3.2.3 Non-MHD Flow.

Ordinary incompressible viscous flow in the absence of external forces is governed by the continuity equation (3.3), the linear momentum equations (3.4) and the vorticity equation (3.6) when $\beta = 0$ is taken in these equations.

3.3 FLOW EQUATIONS IN MARTIN'S FORM.

The equation of continuity (3.3) implies the existence of a streamfunction $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \quad (3.8)$$

We take $\phi(x, y) = \text{constant}$ to be some arbitrary family of curves which generates with the streamlines $\psi(x, y) = \text{constant}$ a curvilinear net so that in the physical plane the independent variables x, y can be replaced by ϕ, ψ .

Having recorded the results from differential geometry in section 2.2, we transform equations (3.3) to (3.7) into a new form with variables ϕ, ψ . These transformations for various classes of steady plane flows have been obtained by Kaloni and Siddiqui [1983] for non-Newtonian second-grade fluid flows, by Chandna and Kaloni [1976] for Cosserat fluid flows and by Chandna, Barron and Chew [1983] for MHD variably-inclined flows.

Equations (3.8), (2.26) and (2.28) give

$$\begin{aligned} \sqrt{E} \cos \alpha &= \frac{\partial x}{\partial \phi} = J \frac{\partial \psi}{\partial y} = Ju = Jq \cos \theta \\ \sqrt{E} \sin \alpha &= \frac{\partial y}{\partial \phi} = -J \frac{\partial \psi}{\partial x} = Jv = Jq \sin \theta \end{aligned}$$

where θ is the direction of the flow in the physical plane. This pair of equations shows that the fluid flows along the streamlines towards higher parameter values of ϕ according as $J > 0$ or $J < 0$. In the following work, we consider that the fluid flows towards higher parameter values of ϕ so that $J = W > 0$.

3.3.1 Infinitely Conducting Flow.

Linear Momentum Equations: Employing (3.8) in the linear momentum equations (3.4) and making use of (2.26), we obtain

$$\frac{\partial h}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial y}{\partial \phi} + \mu \left(-\frac{\partial \omega}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial \omega}{\partial \psi} \frac{\partial x}{\partial \phi} \right) - (\rho \omega - \mu^* \beta \Omega) \frac{\partial y}{\partial \phi} = 0 \quad (3.9)$$

$$\frac{\partial h}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial x}{\partial \phi} + \mu \left(\frac{\partial \omega}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi} \frac{\partial y}{\partial \phi} \right) - (\rho \omega - \mu^* \beta \Omega) \frac{\partial x}{\partial \phi} = 0 \quad (3.10)$$

Multiplying (3.9) by $\frac{\partial x}{\partial \phi}$, (3.10) by $\frac{\partial y}{\partial \phi}$ and subtracting, we get

$$J \frac{\partial h}{\partial \phi} = \mu \left[F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right]$$

Multiplying (3.9) by $\frac{\partial x}{\partial \psi}$, (3.10) by $\frac{\partial y}{\partial \psi}$ and subtracting, we have

$$J \frac{\partial h}{\partial \psi} = \mu \left[-F \frac{\partial \omega}{\partial \psi} + G \frac{\partial \omega}{\partial \phi} \right] - (\rho \omega - \mu^* \beta \Omega) J$$

Solenoidal Equation: Employing (3.8) in the solenoidal equation (3.5) and transforming the resulting equation to (ϕ, ψ) -net, we get

$$\frac{\partial \psi}{\partial y} \left[\frac{\partial \beta}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial \beta}{\partial \psi} \frac{\partial \psi}{\partial x} \right] - \frac{\partial \psi}{\partial x} \left[\frac{\partial \beta}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial \beta}{\partial \psi} \frac{\partial \psi}{\partial y} \right] = 0$$

which implies that

$$\frac{\partial \beta}{\partial \phi} = 0 \quad \text{or} \quad \beta = \beta(\psi) \quad (3.11)$$

Current Density Equation: Transforming the current density equation into the (ϕ, ψ) -net, we have

$$\Omega = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi}$$

Equations of Continuity and Vorticity: Martin [1971] obtained the necessary and sufficient conditions for the flow of a fluid along the coordinate lines $\psi = \text{constant}$ of a curvilinear coordinate system (2.22), with ds^2 given by (2.23), to satisfy the principle of conservation of mass to be

$$Wq = \sqrt{E}, \quad u + iv = \frac{\sqrt{E}}{W} e^{i\alpha}$$

He has also proven that the vorticity equation takes the form

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right]$$

Summing up the above results, we have:

Theorem 3.1. *If the streamlines $\psi(x, y) = \text{constant}$ and an arbitrary family of curves $\phi(x, y) = \text{constant}$ generate a curvilinear net in the physical plane of a steady plane viscous incompressible and infinitely conducting MHD aligned fluid flow, then the flow in independent variables ϕ, ψ is governed by the following system:*

$$\begin{aligned}
q &= \frac{\sqrt{E}}{J} & (\text{continuity}) \\
J \frac{\partial h}{\partial \phi} &= \mu \left[F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right] & (\text{linear} \\
J \frac{\partial h}{\partial \psi} &= \mu \left[-F \frac{\partial \omega}{\partial \psi} + G \frac{\partial \omega}{\partial \phi} \right] - (\rho \omega - \mu^* \beta \Omega) J & (\text{momentum}) \\
\omega &= \frac{1}{J} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \right) \right] & (\text{vorticity}) \\
\Omega &= \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} & (\text{current density}) \\
\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) &= 0 & (\text{Gauss}) \\
\frac{\partial \beta}{\partial \phi} &= 0 & (\text{solenoidal})
\end{aligned} \tag{3.12}$$

of seven equations in eight unknown functions $E, F, G, h, \Omega, \omega, q$ and β of ϕ, ψ .

Given a solution of system (3.12), the pressure function is determined from

$$p = h - \frac{1}{2} \rho \frac{E}{J^2}$$

Furthermore, the flow in the physical plane and hodograph plane is described by

$$\begin{aligned}
\alpha &= \int \frac{J}{E} \{ \Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi \} \\
z &= \int \frac{1}{\sqrt{E}} e^{i\alpha} \{ E d\phi + (F + iJ) d\psi \}
\end{aligned} \tag{3.13}$$

and

$$u + iv = \frac{\sqrt{E}}{J} e^{i\alpha}$$

to complete the solution.

Here $J = W = \sqrt{EG - F^2}$ since ϕ increases in the direction of flow. System (3.12) is underdetermined due to the arbitrariness inherent in the choice of the coordinate lines $\phi = \text{constant}$.

We use the integrability condition $\frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi}$ to eliminate h from the linear momentum equations in (3.12) and we note that the six functions $E(\phi, \psi)$, $F(\phi, \psi)$, $G(\phi, \psi)$, $\omega(\phi, \psi)$, $\Omega(\phi, \psi)$ and $\beta(\psi)$ satisfy the following system:

$$\omega = \frac{1}{J} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \right) \right] \quad (3.14)$$

$$\Omega = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} \quad (3.15)$$

$$\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) = 0 \quad (3.16)$$

$$\mu \left[\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial \omega}{\partial \phi} - \frac{F}{J} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} - \frac{F}{J} \frac{\partial \omega}{\partial \phi} \right) \right] - \rho \frac{\partial \omega}{\partial \phi} + \mu^* \beta \frac{\partial \Omega}{\partial \phi} = 0 \quad (3.17)$$

$$\beta = \beta(\psi) \quad (3.18)$$

of five equations.

3.3.2 Finitely Conducting Flow.

Since a finitely conducting MHD aligned flow is governed by equations (3.3) to (3.7) with $\Omega = \Omega_0$, where Ω_0 is constant, then we have the following theorem:

Theorem 3.2. *If the streamlines $\psi(x, y) = \text{constant}$ of a viscous, incompressible finitely conducting MHD aligned flow are chosen as one set of coordinate curves in a curvilinear coordinate system ϕ, ψ in the physical plane, then the flow is governed by system (3.12) of theorem 3.1 when the function Ω is replaced by a constant Ω_0 in this system and*

$$\Omega_0 = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} \quad (3.19)$$

Employing the integrability condition $\frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi}$ for these flows, we have

$$\mu \left[\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial \omega}{\partial \phi} - \frac{F}{J} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} - \frac{F}{J} \frac{\partial \omega}{\partial \phi} \right) \right] - \rho \frac{\partial \omega}{\partial \phi} = 0 \quad (3.20)$$

where ω is given by (3.14).

3.3.3 Non-MHD Flow.

Ordinary viscous flow is governed by the system of equations (3.3) to (3.7) with $\Omega = \beta = 0$. Equation (3.20) is also satisfied by these flows with ω given by (3.12).

Theorems 3.1 and 3.2 have been used previously to study flows for which certain conditions are placed a priori on the functions E , F and G . We, however, use these theorems for obtaining exact solutions.

3.4 METHOD.

For infinitely conducting flows, equations (3.12) to (3.18) form an underdetermined system of five equations in six unknowns and the reason is the arbitrariness inherent in the choice of the coordinate lines $\phi = \text{constant}$. For the same reason, equations (3.12), (3.16), (3.18) to (3.20) governing finitely conducting flows and equations (3.12), (3.16), (3.20) governing non-MHD flows form underdetermined systems. These systems can be made determinate in a number of different ways. One plausible way is to assume $\phi(x, y) = x$ so that the coordinate system to be dealt with is the von Mises net (x, ψ) . In this case, $E = 1 + \frac{F^2}{G}$ and the underdetermined systems reduce to determinate systems.

To analyze whether a given family of curves $\frac{y - f(x)}{g(x)} = \text{constant}$ can or cannot be the streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{y - f(x)}{g(x)} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (3.21)$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$.

Employing (3.21) and $\phi = x$ in (2.24) and (2.26), we have

$$\begin{aligned} E &= 1 + [f'(x) + g'(x)\gamma(\psi)]^2, \quad G = g^2(x)\gamma'^2(\psi) \\ F &= [f'(x) + g'(x)\gamma(\psi)]g(x)\gamma'(\psi), \quad J = W = g(x)\gamma'(\psi) \end{aligned} \quad (3.22)$$

3.4.1 Infinitely Conducting Flow.

Using (3.22) and $\phi = x$ in (3.12) to (3.18), Gauss equation (3.16) is identically satisfied and we have the following theorem:

Theorem 3.3. *If a steady, plane, viscous, incompressible, electrically conducting fluid of infinite electrical conductivity flows along $\frac{y - f(x)}{g(x)} = \text{constant}$, in the presence of an aligned magnetic field, then the functions $f(x)$, $g(x)$, $\beta(\psi)$ and $\gamma(\psi)$ must satisfy*

$$g(x)\gamma'(\psi)\frac{\partial^2\omega}{\partial x^2} - 2[f'(x) + g'(x)\gamma(\psi)]\frac{\partial^2\omega}{\partial x\partial\psi} + \frac{1 + [f'(x) + g'(x)\gamma(\psi)]^2}{g(x)\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2}$$

$$\begin{aligned}
& + \left\{ \left[\frac{2f'(x)g'(x)}{g(x)} - f''(x) \right] + \left[\frac{2g'^2(x)}{g(x)} - g''(x) \right] \gamma(\psi) - \frac{1 + f'^2(x)}{g(x)} \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right. \\
& \left. - \frac{2f'(x)g'(x)}{g(x)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{g'^2(x)}{g(x)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right\} \frac{\partial \omega}{\partial \psi} - \frac{\rho}{\mu} \frac{\partial \omega}{\partial x} + \frac{\mu^*}{\mu} \beta(\psi) \frac{\partial \Omega}{\partial x} = 0 \quad (3.23)
\end{aligned}$$

where ω and Ω are given by

$$\begin{aligned}
\omega = & \left[\frac{f''(x)}{g(x)} - \frac{2f'(x)g'(x)}{g^2(x)} \right] \frac{1}{\gamma'(\psi)} + \left[\frac{g''(x)}{g(x)} - \frac{2g'^2(x)}{g^2(x)} \right] \frac{\gamma(\psi)}{\gamma'(\psi)} \\
& + \frac{1 + f'^2(x)}{g^2(x)} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2f'(x)g'(x)}{g^2(x)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{g'^2(x)}{g^2(x)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} \quad (3.24)
\end{aligned}$$

$$\Omega = \beta\omega - \frac{1 + [f'(x) + g'(x)\gamma(\psi)]^2}{g^2(x)\gamma'^2(\psi)} \beta'(\psi) \quad (3.25)$$

and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

A given family of curves $\frac{y - f(x)}{g(x)} = \text{constant}$ is a permissible family of streamlines only if the solution obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$. Having established a streamline pattern, exact integral for the flow pattern is then obtained.

The two definitions of an exact integral of the flow governed by system (3.12) are:

(i) a set of functions u, v, h, ω, Ω and β constitutes an exact integral of system (3.17) if this system is satisfied by these functions for all x, y and for all values of ρ, μ and μ^* .

(ii) a set of functions u, v, h, ω, Ω and β constitutes an exact integral of system (3.17) if system (3.17) and the boundary conditions of a realistically imposed problem are satisfied by these functions for all x, y and for all values of ρ, μ and μ^* .

3.4.2 Finitely Conducting Flow.

Using (3.22) and $\phi = x$ in (3.12), (3.16), (3.18) to (3.20), we find that Gauss equation (3.16) is identically satisfied and we have the following theorem:

Theorem 3.4. *If a steady, plane, viscous, incompressible, electrically conducting fluid of finite electrical conductivity flows along $\frac{y - f(x)}{g(x)} = \text{constant}$, in the presence of an aligned magnetic field, then the functions $f(x)$, $g(x)$, $\beta(\psi)$ and $\gamma(\psi)$ must satisfy*

$$g(x)\gamma'(\psi)\frac{\partial^2\omega}{\partial x^2} - 2[f'(x) + g'(x)\gamma(\psi)]\frac{\partial^2\omega}{\partial x\partial\psi} + \frac{1 + [f'(x) + g'(x)\gamma(\psi)]^2}{g(x)\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} \\ \left\{ \left[\frac{2f'(x)g'(x)}{g(x)} - f''(x) \right] + \left[\frac{2g'^2(x)}{g(x)} - g''(x) \right] \gamma(\psi) - \frac{1 + f'^2(x)}{g(x)} \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right. \\ \left. - \frac{2f'(x)g'(x)}{g(x)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{g'^2(x)}{g(x)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right\} \frac{\partial\omega}{\partial\psi} - \frac{\rho}{\mu} \frac{\partial\omega}{\partial x} = 0 \quad (3.26)$$

$$\beta(\psi)\omega - \frac{1 + [f'(x) + g'(x)\gamma(\psi)]^2}{g^2(x)\gamma'^2(\psi)}\beta'(\psi) = \Omega_0 \quad (3.27)$$

where ω is given by equation (3.24) and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

3.4.3 Non-MHD Flow.

Employing (3.22) and $\phi = x$ in (3.12), (3.16) and (3.20), Gauss equation (3.16) is identically satisfied and we have:

Theorem 3.5. *If a steady plane viscous incompressible non-conducting fluid flows along $\frac{y - f(x)}{g(x)} = \text{constant}$, then the functions $f(x)$, $g(x)$, $\beta(\psi)$ and $\gamma(\psi)$ must satisfy equation (3.26) with ω given by (3.24).*

In the next section, we determine exact integrals as defined by the above two definitions for infinitely conducting MHD aligned flows, finitely conducting MHD aligned flows and non-MHD flows. Interchanging the roles of x and y , we can study the family of curves $\frac{x - k(y)}{m(y)} = \text{constant}$ in the same manner.

3.5 APPLICATIONS.

3.5.1 Example I. (Parabolic flows along $y - m_1 x^2 - m_2 x = \text{constant}$).

In this example, we pose our two questions for the family of curves $y - m_1 x^2 - m_2 x = \text{constant}$ when m_1, m_2 are two real constants. If the answer to our first question is in the affirmative, we have

$$y - m_1 x^2 - m_2 x = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (3.28)$$

where $\gamma(\psi)$ is an arbitrary function of ψ .

Comparing (3.28) with (3.21), we have

$$f(x) = m_1 x^2 + m_2 x, \quad g(x) = 1 \quad (3.29)$$

The streamline pattern for this flow is shown in Figure 3.1.

Infinitely Conducting Flow

Using (3.29), (3.24) and (3.25), equation (3.23) reduces to

$$\sum_{\substack{n=0 \\ n \neq 3}}^4 A_n(\psi) (2m_1 x + m_2)^n = 0 \quad (3.30)$$

where

$$\begin{aligned} A_0(\psi) &= A_4(\psi) - 4m_1 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 12m_1^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ A_1(\psi) &= -\frac{4m_1}{\mu} \left\{ (\rho - \mu^* \beta^2) \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ A_2(\psi) &= 2A_4(\psi) - 12m_1 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\ A_4(\psi) &= \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' \end{aligned}$$

Equation (3.30) is a fourth degree polynomial in $(2m_1 x + m_2)$ with coefficients as functions of ψ only. Since x, ψ are independent variables, it follows that equation

(3.30) can hold true if all the coefficients of this polynomial vanish simultaneously and we have

$$A_4(\psi) = A_2(\psi) = A_1(\psi) = A_0(\psi) = 0 \quad (3.31)$$

Using the consequence of $A_4(\psi) = 0$, $A_2(\psi) = 0$ in $A_0(\psi) = 0$, $A_1(\psi) = 0$, we find that

$$\gamma(\psi) = a_1\psi + a_2, \quad \beta(\psi) = \beta_0 \quad (3.32)$$

where $a_1 \neq 0$, a_2 and $\beta_0 \neq 0$ are arbitrary constants.

Thus, the family of curves $y - (m_1x^2 + m_2x) = \text{constant}$ are permissible streamlines for the fluid flow under consideration and equation (3.28) takes the form

$$y - m_1x^2 - m_2x = a_1\psi(x, y) + a_2 \quad (3.33)$$

Using (3.32) and (3.33) in (3.8), (3.1), (3.2) and (3.4) to (3.7), we find that the exact integral for our flow problem is given by

$$\begin{aligned} u &= \frac{1}{a_1}, \quad v = \frac{1}{a_1} (2m_1x + m_2), \quad H_1 = \frac{\beta_0}{a_1}, \quad H_2 = \frac{\beta_0}{a_1} (2m_1x + m_2) \\ p &= \frac{2m_1}{a_1^2} (\mu^* \beta_0^2 - \rho) y - \frac{2m_1 \mu^* \beta_0^2}{a_1^2} (m_1x^2 + m_2x) - \frac{\rho}{2a_1^2} (1 + m_2^2) + p_0 \\ \omega &= \frac{2m_1}{a_1}, \quad \Omega = \frac{2m_1}{a_1} \beta_0 \end{aligned} \quad (3.34)$$

where p_0 is an arbitrary constant.

Finitely Conducting Flow

Using (3.29) and (3.24) in (3.26) and (3.27), we obtain the following equations to be satisfied by $\gamma(\psi)$ and $\beta(\psi)$:

$$\begin{aligned} \sum_{n=0}^4 B_n(\psi) (2m_1x + m_2)^n &= 0 \\ n &\neq 3 \end{aligned} \quad (3.35)$$

and

$$\sum_{n=0}^2 C_n(\psi) (2m_1x + m_2)^n = 0 \quad (3.36)$$

$$n \neq 1$$

where

$$\begin{aligned} B_0(\psi) &= B_4(\psi) - 4m_1 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 12m_1^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ B_1(\psi) &= -\frac{4m_1\rho}{\mu} \left\{ \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\} \\ B_2(\psi) &= 2B_4(\psi) - 12m_1 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\ B_4(\psi) &= \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' \\ C_0(\psi) &= \frac{2m_1\gamma'^2(\psi)\beta(\psi) + \gamma''(\psi)\beta(\psi) - \gamma'(\psi)\beta'(\psi)}{\gamma'^3(\psi)} - \Omega_0 \\ C_2(\psi) &= \frac{\gamma''(\psi)\beta(\psi) - \gamma'(\psi)\beta'(\psi)}{\gamma'^3(\psi)} \end{aligned}$$

Since x, ψ are independent variables and equations (3.35) and (3.36) hold true in the entire (x, ψ) -plane, it follows that

$$B_0(\psi) = B_1(\psi) = B_2(\psi) = B_4(\psi) = C_0(\psi) = C_2(\psi) = 0 \quad (3.37)$$

Requiring equations (3.37) to hold true simultaneously, we have

$$\gamma(\psi) = b_1\psi + b_2, \quad \beta(\psi) = \frac{b_1\Omega_0}{2m_1} \quad (3.38)$$

Employing (3.38) in (3.28) and using (3.8), (3.1), (3.2) and (3.4) to (3.7), we find that

$$\begin{aligned} u &= \frac{1}{b_1}, \quad v = \frac{1}{b_1} (2m_1x + m_2), \quad H_1 = \frac{\Omega_0}{2m_1}, \quad H_2 = \frac{\Omega_0}{2m_1} (2m_1x + m_2), \\ p &= \left(\frac{\mu^*\Omega_0^2}{2m_1} - \frac{2m_1\rho}{b_1^2} \right) y - \frac{\mu^*\Omega_0^2}{2m_1} (m_1x^2 + m_2x) - \frac{\rho}{2b_1^2} (1 + m_2^2) + p_0 \\ \omega &= \frac{2m_1}{b_1}, \quad \Omega = \Omega_0 \end{aligned} \quad (3.39)$$

Non-MHD Flow

The exact integral for ordinary viscous non-MHD flow is given by (3.39) with $\Omega_0 = 0$.

Therefore, we have:

Theorem 3.6. *Streamline pattern $y - m_1 x^2 - m_2 x = \text{constant}$ of steady plane motion is permissible for infinitely conducting MHD aligned, finitely conducting MHD aligned and non-MHD flow with solutions given by (3.34), (3.39) and (3.39) with $\Omega_0 = 0$ respectively.*

The parabolic flow that we have studied is a generalized Beltrami flow since $\text{curl}(\omega \times \mathbf{V}) = 0$ is satisfied by the obtained exact integral. Wang [1991] has given an excellent review of these flows for the Navier-Stokes equations.

We consider the physical problem of fluid flow impinging on a porous wall parallel to y-axis at $x = x_0$ with constant suction velocity $(u_0, 0)$ and external(outside) magnetic field $(H_0, 0)$. Using these conditions and equations (2.65) and (2.66), we obtain the exact integral for our physical problem given by

$$\begin{aligned} u(x, y) &= u_0, & v(x, y) &= 2m_1 u_0 (x - x_0), \\ H_1(x, y) &= H_0, & H_2(x, y) &= 2m_1 H_0 (x - x_0), \\ p(x, y) &= 2m_1 (\mu^* H_0^2 - \rho u_0^2) y - 2m_1 \mu^* H_0^2 (x^2 - 2x_0 x) - \frac{\rho}{2} u_0^2 (1 + 4m_1^2 x_0^2) + p_0, \\ \omega(x, y) &= 2m_1 u_0, & \Omega(x, y) &= 2m_1 H_0 \end{aligned} \tag{3.40}$$

This exact solution of our physical problem satisfies both infinitely conducting and finitely conducting MHD aligned flow equations when the streamline pattern in the flow region is given by

$$y - m_1 x^2 + 2m_1 x_0 x = \text{constant} \tag{3.41}$$

The streamline pattern for this flow is given in Figure 3.2.

If $m_1 = 0$ is taken in equation (3.28) and we proceed as outlined in parabolic flows above, then $\gamma(\psi)$ must satisfy

$$y - m_2 x = \gamma(\psi), \quad \frac{1}{\gamma'(\psi)} = c_1 (y - m_2 x)^2 + c_2 (y - m_2 x) + c_3,$$

and

$$\begin{aligned} 6\psi &= 2c_1 \gamma^3(\psi) + 3c_2 \gamma^2(\psi) + 6c_3 \gamma(\psi) + c_4 \\ &= 2c_1 (y - m_2 x)^3 + 3c_2 (y - m_2 x)^2 + 6c_3 (y - m_2 x) + c_4 \end{aligned} \quad (3.42)$$

for both finitely conducting and infinitely conducting cases when $c_1 \neq 0$, c_2 , c_3 and c_4 are arbitrary constants. However, whereas $\beta(\psi)$ remains an arbitrary function for infinitely conducting case, it is given by

$$\beta(\psi) = \left[c_3 - \frac{\Omega_0}{1 + m_2^2} \gamma(\psi) \right] \gamma'(\psi) = \frac{(1 + m_2^2) c_3 - \Omega_0 (y - m_2 x)}{(1 + m_2^2) [c_1 (y - m_2 x)^2 + c_2 (y - m_2 x) + c_3]} \quad (3.43)$$

for the finitely conducting case. Following the approach taken before, we have the theorem:

Theorem 3.7. *Steady plane parallel flows with $y - m_2 x = \text{constant}$ as the streamlines are permissible for infinitely conducting MHD aligned, finitely conducting MHD aligned and non-MHD fluid flows. Solutions for infinitely conducting fluid are given by*

$$\begin{aligned} u &= c_1 (y - m_2 x)^2 + c_2 (y - m_2 x) + c_3, \quad v = m_2 u \\ H_1 &= \beta(\psi) u, \quad H_2 = \beta(\psi) v \\ p &= p_0 + 2\mu c_1 (1 + m_2^2) (x + m_2 y) - \frac{1}{2} (1 + m_2^2) \mu^* \beta^2(\psi) u^2 \\ \omega &= - (1 + m_2^2) [2c_1 (y - m_2 x) + c_2] \\ \Omega &= \beta(\psi) \omega - (1 + m_2^2) [c_1 (y - m_2 x)^2 + c_2 (y - m_2 x) + c_3]^2 \beta'(\psi) \end{aligned} \quad (3.44)$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is an arbitrary function of ψ and $\psi(x, y)$ is

given by equation (3.43). Solutions for finitely conducting fluid are given by

$$\begin{aligned}
u &= c_1 (y - m_2 x)^2 + c_2 (y - m_2 x) + c_3, & v &= m_2 u \\
H_1 &= c_5 - \frac{\Omega_0}{1 + m_2^2} (y - m_2 x), & H_2 &= m_2 c_5 - \frac{m_2 \Omega_0}{1 + m_2^2} (y - m_2 x) \\
p &= \bar{p}_0 + 2\mu c_1^2 (1 + m_2^2) (x + m_2 y) + \mu^* c_3 \Omega_0 (y - m_2 x) \\
&\quad - \frac{\mu^* \Omega_0^2}{2(1 + m_2^2)} (y - m_2 x)^2 \\
\omega &= - (1 + m_2^2) [c_2 + 2c_1 (y - m_2 x)], & \Omega &= \Omega_0
\end{aligned} \tag{3.45}$$

where \bar{p}_0 is an arbitrary constant. For non-MHD fluids, the solutions are given by equations (3.45) with $\Omega_0 = 0$ and $c_5 = 0$.

Various realistically imposed physical problems for MHD parallel and non-MHD parallel flows have been studied by researchers in the past and these are well documented in texts (c.f. Pai [1962], Schlichting [1968]).

3.5.2 Example II. (Flow along quartics $\frac{y - ax^4}{x} = \text{constant}$).

Following the steps taken in the previous example, we obtain

$$\left. \begin{aligned} \frac{y - ax^4}{x} &= \gamma(\psi), & \gamma'(\psi) &\neq 0 \\ f(x) &= ax^4, & g(x) &= x \end{aligned} \right\} \tag{3.46}$$

The streamline pattern for this flow is shown in Figure 3.3.

Infinitely Conducting Flow

Employing (3.46), (3.24) and (3.25) in equation (3.23), we get

$$\sum_{n=0}^4 D_{3n}(\psi) x^{3n} = 0 \tag{3.47}$$

where

$$\begin{aligned}
D_0(\psi) &= 12 [1 + 3\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - 24\gamma(\psi) + 12 [1 + \gamma^2(\psi)] \gamma(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
&\quad + [1 + \gamma^2(\psi)]^2 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' + \frac{2\mu^*}{\mu} [1 + \gamma^2(\psi)] \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{2}{\mu} [\rho - \mu^* \beta^2(\psi)] \left\{ [1 + \gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{2\gamma(\psi)}{\gamma'(\psi)} \right\} \\
D_3(\psi) &= 96a \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - 24a + 24a [1 + 3\gamma^2(\psi)] \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
&\quad + 16a [1 + \gamma^2(\psi)] \gamma(\psi) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' - \frac{8\mu^* a}{\mu} \frac{\gamma(\psi)\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\
&\quad - \frac{4}{\mu} [\rho - \mu^* \beta^2(\psi)] a \left[\frac{2\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{1}{\gamma'(\psi)} \right] \\
D_6(\psi) &= 240a^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 32 [1 + 3\gamma^2(\psi)] a^2 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' \\
&\quad - \frac{64}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] a^2 \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* a^2 \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
D_9(\psi) &= 256a^3 \gamma(\psi) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' - 384a^3 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
\text{and} \\
D_{12}(\psi) &= 256a^4 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}'
\end{aligned}$$

Since equation (3.47) is to hold true identically, it follows that $D_0(\psi)$, $D_3(\psi)$, $D_6(\psi)$, $D_9(\psi)$ and $D_{12}(\psi)$ vanish simultaneously. This requirement yields

$$\beta(\psi) = \beta_0, \quad \frac{1}{6\mu} [\mu^* \beta_0^2 - \rho] \psi + \psi_0 = \frac{y - ax^4}{x}$$

where ψ_0 and β_0 are arbitrary constants and $\mu^* \beta_0^2 \neq \rho$. Exact integral for this

infinitely conducting MHD aligned flow along quartics is obtained and we have

$$\begin{aligned}
u &= \frac{6\mu}{\mu^*\beta_0^2 - \rho} \frac{1}{x}, & v &= \frac{6\mu}{\mu^*\beta_0^2 - \rho} \frac{3ax^4 + y}{x^2} \\
H_1 &= \beta_0 u, & H_2 &= \beta_0 v \\
p &= p_0 + \frac{6\mu^2}{(\mu^*\beta_0^2 - \rho)^2} \left[\mu^*\beta_0^2 \frac{42ax^4y - 27a^2x^8 - x^2 - 3y^2}{x^4} - 2\rho \frac{30ax^2y + 1}{x^2} \right] \\
\omega &= \frac{6\mu}{\mu^*\beta_0^2 - \rho} \frac{6ax^4 - 2y}{x^3}, & \Omega &= \beta_0 \omega
\end{aligned} \tag{3.48}$$

where p_0 is an arbitrary constant.

Finitely Conducting and non-MHD Flows

Using (3.46) and (3.24) in (3.26) and (3.27), we have

$$\sum_{n=0}^4 E_{3n}(\psi) x^{3n} = 0 \tag{3.49}$$

and

$$\sum_{n=0}^6 F_n(\psi) x^n = 0 \tag{3.50}$$

where

$$\begin{aligned}
E_0(\psi) &= 12 [1 + 3\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - 24\gamma(\psi) + 12 [1 + \gamma^2(\psi)] \gamma(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
&\quad + [1 + \gamma^2(\psi)]^2 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' + \frac{2\rho}{\mu} \left\{ [1 + \gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{2\gamma(\psi)}{\gamma'(\psi)} \right\} \\
E_3(\psi) &= 96a \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - 24a + 24a [1 + 3\gamma^2(\psi)] \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
&\quad + 16a [1 + \gamma^2(\psi)] \gamma(\psi) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' - \frac{4a\rho}{\mu} \left[\frac{2\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{1}{\gamma'(\psi)} \right] \\
E_6(\psi) &= 240a^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 32 [1 + 3\gamma^2(\psi)] a^2 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' - \frac{64\rho a^2}{\mu} \left\{ \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\} \\
E_9(\psi) &= 256a^3 \gamma(\psi) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' - 384a^3 \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
E_{12}(\psi) &= 256a^4 \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}' \\
F_1(\psi) &= F_4(\psi) = F_5(\psi) = 0
\end{aligned}$$

$$F_0(\psi) = \left\{ \left[\frac{[1 + \gamma^2(\psi)] \gamma''(\psi) - 2\gamma(\psi)\gamma'(\psi)}{\gamma'^3(\psi)} \right] \beta(\psi) - \left[\frac{1 + \gamma^2(\psi)}{\gamma'^2(\psi)} \right] \beta'(\psi) \right\}$$

$$F_2(\psi) = \Omega_0$$

$$F_3(\psi) = 4a \left\{ \left[\frac{2\gamma(\psi)\gamma''(\psi) + \gamma'^2(\psi)}{\gamma'^3(\psi)} \right] \beta(\psi) - \frac{2\gamma(\psi)}{\gamma'^2(\psi)} \beta'(\psi) \right\}$$

and

$$F_6(\psi) = 16a^2 \left\{ \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \beta(\psi) - \frac{1}{\gamma'^2(\psi)} \beta'(\psi) \right\}$$

Since x, ψ are independent variables, it follows that all coefficients (functions of ψ) of polynomials (3.49) and (3.50) in variable x must vanish simultaneously. This requirement yields

$$\Omega_0 = 0, \quad \gamma''(\psi) = 0 \quad \text{with} \quad \gamma'(\psi) = -\frac{\rho}{6\mu} \quad \text{and} \quad \beta(\psi) = 0.$$

Therefore, finitely conducting MHD aligned flow along the quartics is not permissible. However, non-MHD ordinary viscous flow is permissible and is given by

$$\frac{y - ax^4}{x} = -\frac{\rho}{6\mu} \psi + \psi_0$$

The exact integral for this flow is

$$\begin{aligned} u &= -\frac{6\mu}{\rho} \frac{1}{x}, & v &= -\frac{6\mu}{\rho} \frac{y + 3ax^4}{x^2} \\ \omega &= \frac{12\mu}{\rho} \frac{y - 3ax^4}{x^3} \\ p &= -\frac{12\mu^2}{\rho} \frac{30ax^2y + 1}{x^2} + p_0 \end{aligned} \tag{3.51}$$

where p_0 is an arbitrary constant.

Summing up, we have:

Theorem 3.8. *Steady plane flow with $\frac{y-ax^4}{x} = \text{constant}$ as a family of streamlines is not allowed by finitely conducting MHD aligned flow but is allowed by infinitely conducting MHD aligned and non-MHD flows with solutions given by (3.48) and (3.51) respectively.*

Flow along quartics yielded the motion for both infinitely conducting MHD and non-MHD cases, such that the streamfunction is linear with respect to y . Flows when the streamfunction is linear with respect to y or x were studied for Navier-Stokes equations by D. Riabouchinsky [1924] and R. Berker [1963].

Considering the physical problem of fluid flow impinging on a porous stretching surface parallel to y -axis at $x = x_0$ with constant suction velocity $(u_0, 0)$, stretching velocity $\left(0, \frac{u_0}{x_0}y + 3au_0x_0^3\right)$ and external magnetic field $\sqrt{\frac{6\mu u_0 + \rho u_0^2 x_0}{\mu^* x_0}} \left(2 - \frac{x}{x_0}, \frac{y}{x_0} + 3ax_0^3\right)$, we use these boundary conditions and equations (2.65), (2.66) to obtain the exact integral for our physical problem which is given by

$$\begin{aligned} u &= u_0 \frac{x}{x_0}, & v &= u_0 x_0 \left(3ax^2 + \frac{y}{x^2}\right) \\ H_1 &= \sqrt{\frac{6\mu + \rho u_0 x_0}{\mu^* u_0 x_0}} u, & H_2 &= \sqrt{\frac{6\mu + \rho u_0 x_0}{\mu^* u_0 x_0}} v \\ \omega &= 2u_0 x_0 \left(\frac{3ax^4 - y}{x^3}\right), & \Omega &= \sqrt{\frac{6\mu + \rho u_0 x_0}{\mu^* u_0 x_0}} \omega \end{aligned}$$

and

$$\begin{aligned} p &= p_0 + \frac{u_0^2 x_0^2}{6} \left\{ \frac{6\mu + \rho u_0 x_0}{u_0 x_0} \left[\frac{42ax^4 y - 27a^2 x^8 - 3y^2 - x^2}{x^4} \right] \right. \\ &\quad \left. - 2\rho \left[\frac{30ax^2 y + 1}{x^2} \right] \right\} \end{aligned}$$

The streamline pattern for this physical problem is shown in Figure 3.4

3.5.3 Example III. (Flow with $x^2 y = \text{constant}$ as streamlines).

We assume that

$$x^2 y = \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (3.52)$$

and, therefore, we have

$$f(x) = 0, \quad g(x) = \frac{1}{x^2} \quad (3.53)$$

The streamline pattern for this flow is shown in Figure 3.5.

Infinitely Conducting Flow

Proceeding as in previous example, we find that $\gamma(\psi)$ and $\beta(\psi)$ must satisfy

$$\sum_{n=0}^4 G_{3n}(\psi) x^{3n} = 0 \quad (3.54)$$

where

$$\begin{aligned} G_{12}(\psi) &= \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}', \\ G_9(\psi) &= \left[\frac{4\mu^* \beta^2(\psi) - 4\rho}{\mu} \right] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{4\mu^*}{\mu \gamma'^2(\psi)} \beta(\psi) \beta'(\psi), \\ G_6(\psi) &= 24 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 36 \gamma(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 8 \gamma^2(\psi) G_{12}(\psi), \\ G_3(\psi) &= \left[\frac{4\mu^* \beta^2(\psi) - 4\rho}{\mu} \right] \left[\frac{\gamma(\psi) \gamma'^2(\psi) - 2 \gamma^2(\psi) \gamma''(\psi)}{\gamma'^3(\psi)} \right] \\ &\quad + \frac{8\mu^* \gamma^2(\psi)}{\mu \gamma'^2(\psi)} \beta(\psi) \beta'(\psi) \end{aligned}$$

and

$$G_0(\psi) = 12 \frac{\gamma^2(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} + 48 \gamma^3(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 16 \gamma^4(\psi) G_{12}(\psi)$$

The polynomial (3.54) in x , with coefficients as functions of ψ only, holds true provided all coefficients vanish simultaneously. This requirement and equation (3.52) yield

$$x^2 y = \gamma(\psi) = b_1 \psi + b_2 \quad \text{and} \quad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}} \quad (3.55)$$

where $b_1 \neq 0$ and b_2 are arbitrary constants.

Proceeding as in previous examples, the exact integral for the infinitely conducting MHD aligned flow with $x^2 y = \text{constant}$ as streamlines, is given by

$$\begin{aligned} u &= \frac{x^2}{b_1}, \quad v = -\frac{2xy}{b_1}, \quad H_1 = \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\ p &= \frac{2\mu}{b_1} x - \frac{\rho}{2b_1^2} x^2 (x^2 + 4y^2) + p_0 \\ \omega &= -\frac{2y}{b_1}, \quad \Omega = -\frac{2}{b_1} \sqrt{\frac{\rho}{\mu^*}} y \end{aligned} \quad (3.56)$$

where p_0 is an arbitrary constant.

Finitely Conducting and non-MHD Flows

We employ (3.53) and (3.24) in (3.26) to obtain the following equation in unknown function $\gamma(\psi)$

$$\sum_{n=0}^4 M_{3n}(\psi) x^{3n} = 0 \quad (3.57)$$

where

$$\begin{aligned} M_{12}(\psi) &= \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \right\}', \\ M_9(\psi) &= -\frac{4\mu}{\rho} \frac{\gamma''(\psi)}{\gamma'^3(\psi)}, \\ M_6(\psi) &= 24 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 36\gamma(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 8\gamma^2(\psi) M_{12}(\psi), \\ M_3(\psi) &= \frac{4\mu}{\rho} \left[\frac{2\gamma^2(\psi)\gamma''(\psi) - \gamma(\psi)\gamma'^2(\psi)}{\gamma'^3(\psi)} \right], \end{aligned}$$

and

$$M_0(\psi) = 12 \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} + 48\gamma^3(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + 16\gamma^4(\psi) M_{12}(\psi)$$

Equation (3.57) must hold true for both finitely conducting and non-MHD flows. Since $M_0(\psi)$, $M_3(\psi)$, $M_6(\psi)$, $M_9(\psi)$ and $M_{12}(\psi)$ are required to vanish simultaneously for (3.57) to hold true, it follows that $\gamma(\psi) = 0$ and this contradicts (3.52). Therefore, the streamline pattern of this example is not permissible for either finitely conducting MHD aligned or non-MHD flows. Summing up, we have the following theorem:

Theorem 3.9. *Steady plane flow with $x^2y = \text{constant}$ as a family of streamlines is only permissible for infinitely conducting MHD aligned flow with exact solution given by (3.56)*

We now deal with the motion of MHD aligned flow of a viscous incompressible fluid of infinite electrical conductivity on the right of an infinite plate at $x = 0$.

The steady motion of fluid against the plate takes place along $x^2y = \text{constant}$ when a magnetic field of intensity $(x^3, -3x^2y)$ is present everywhere before the motion starts. With this configuration of the physical problem, a two-dimensional flow with stagnation line $x = 0$ is achieved. This flow satisfies the flow equations, the mechanical boundary conditions and the continuity conditions of the normal and tangential components of the magnetic field. The exact integral of this flow is given by (3.56) where b_1 is any constant. Figure 3.6 shows the streamline pattern for this stagnation point flow.

3.5.4 Example IV. (Flow with $ye^z = \text{constant}$ as streamlines).

In this example, we prove that electrically conducting fluid of finite electrical conductivity cannot flow along the family of curves $ye^z = \text{constant}$. To prove this claim, we assume the contrary to arrive at a contradiction. We assume that

$$ye^z = \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (3.58)$$

and we have

$$f(x) = 0, \quad g(x) = e^{-x} \quad (3.59)$$

Proceeding as above, we have

$$\sum_{\substack{n=0 \\ n \neq 1}}^4 A_n(\psi) e^{nz} = 0 \quad (3.60)$$

where

$$A_0(\psi) = \left[\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\gamma(\psi)}{\gamma'(\psi)} \right]' \left[\gamma(\psi) - \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} \right] + \frac{\gamma^2(\psi)}{\gamma'(\psi)} \left[\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\gamma(\psi)}{\gamma'(\psi)} \right]''$$

$$\begin{aligned}
A_2(\psi) &= 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 4\gamma(\psi) \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' + \frac{1}{\gamma'(\psi)} \left[\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\gamma(\psi)}{\gamma'(\psi)} \right]'' \\
&\quad + \frac{\gamma^2(\psi)}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]'' + \left[\gamma(\psi) - \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right] \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \\
&\quad - \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \left[\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\gamma(\psi)}{\gamma'(\psi)} \right]' \\
A_3(\psi) &= -\frac{2}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\mu^*}{\mu} \frac{1}{\gamma'^2(\psi)} \beta(\psi) \beta'(\psi) \\
A_4(\psi) &= \frac{1}{\gamma'(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]'' - \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \left[\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]'
\end{aligned}$$

Since x, ψ are independent variables and $\{e^{nx}, n = 0, 2, 3, 4\}$ is a linearly independent set, it follows from (3.60) that the coefficients of e^{nx} , $n = 0, 2, 3, 4$ must vanish simultaneously. Taking these four coefficients to be zero, we arrive at the conclusion that $\gamma(\psi) = 0$. However, $\gamma'(\psi) \neq 0$ was assumed and we have arrived at a contradiction. Therefore, electrically conducting fluid of infinite electrical conductivity cannot flow along the family of curves $ye^x = \text{constant}$. Similarly, we can show that finitely conducting and non-MHD fluids cannot flow along this family of curves.

Proceeding as in the last example, we can prove that none of the three types of fluid motions considered is possible for many streamline patterns that belong to the form $\frac{y - f(x)}{g(x)} = \text{constant}$. Some examples are: $y - x^n = \text{constant}$ when $n \in N$ and $n \geq 4$, $y - \sin x = \text{constant}$, $y - xe^x = \text{constant}$, $y - x - \frac{1}{x} = \text{constant}$, $y - x^{3/2} = \text{constant}$, $y - x^{2/3} = \text{constant}$, $y - x^2 e^{x^2} = \text{constant}$ and $y - \ln x = \text{constant}$.

More exact solutions for infinitely conducting aligned MHD flow are outlined in the following examples. The solutions for $\gamma(\psi)$ and $\beta(\psi)$ are first obtained followed by the exact integrals for the unbounded domain. The boundary conditions of realistically imposed physical problems and the determination of the arbitrary constants are then given.

3.4.5 Example V. (Flow with $y - e^{mx} - m_1 x^2 - m_2 x = \text{constant}$).

We assume that

$$y - e^{mz} - m_1 x^2 - m_2 x = \gamma(\psi), \quad m \neq 0$$

and find that

$$\gamma(\psi) = \frac{1}{\mu m} [\rho - \mu^* \beta_0^2] \psi + \psi_0$$

$$\beta(\psi) = \beta_0, \quad \mu^* \beta_0^2 \neq \rho$$

$$u = \frac{\mu m}{[\rho - \mu^* \beta_0^2]}, \quad v = \frac{\mu m}{[\rho - \mu^* \beta_0^2]} [m e^{mz} + 2m_1 x + m_2]$$

$$H_1 = \beta_0 u, \quad H_2 = \beta_0 v$$

$$p = p_0 - \frac{\mu^2 m^2}{2 [\rho - \mu^* \beta_0^2]^2} \left\{ \mu^* \beta_0^2 [m^2 e^{2mz} + 4m_1^2 x^2 + 4mm_1 x e^{mz} + 2mm_2 e^{mz} + 4m_1 m_2 x] + 4m_1 [\rho - \mu^* \beta_0^2] y \right\}$$

The boundary conditions and the constants are given by

$$u(0, y) = u_0 \neq 0, \quad v(0, y) = 0$$

$$\tilde{H}_{\text{out}} = (H_0, 0)$$

$$m = \frac{\rho u_0^2 - \mu^* H_0^2}{\mu u_0}$$

$$m_2 = -m, \quad \beta_0 = \frac{H_0}{u_0}$$

and m_1, p_0 are arbitrary constants. The flow patterns for the unbounded domain and for the boundary value problem are given in Figures 3.7 and 3.8 respectively.

3.4.6 Example VI. (Flow with $y(1 + m_1 e^{az}) - m_2 e^{az} = \text{constant}$ as streamlines).

We let

$$y(1 + m_1 e^{az}) - m_2 e^{az} = \gamma(\psi)$$

and we obtain

$$\begin{aligned}
\gamma(\psi) &= \frac{1}{\mu a} [\rho - \mu^* \beta_0^2] \psi + \psi_0 \\
\beta(\psi) &= \beta_0, \quad \mu^* \beta_0^2 \neq \rho \\
u &= \frac{\mu a}{[\rho - \mu^* \beta_0^2]} (1 + m_1 e^{ax}) \\
v &= \frac{\mu a^2}{[\rho - \mu^* \beta_0^2]} [m_2 - m_1 y] e^{ax} \\
H_1 &= \beta_0 u, \quad H_2 = \beta_0 v \\
p &= p_0 - \frac{a^2 \mu^2}{2 [\rho - \mu^* \beta_0^2]^2} \left\{ \mu^* \beta_0^2 \left[2m_1 e^{ax} \right. \right. \\
&\quad \left. \left. + a^2 (m_2 - m_1 y)^2 e^{2ax} \right] + \rho (1 + m_1^2 e^{2ax}) \right\}
\end{aligned}$$

The boundary conditions and the constants are

$$\begin{aligned}
u(0, y) &= 0, \quad v(0, y) = Ay + B, \quad A \neq 0 \\
\tilde{H}_{out} &= \left(H_0 x, -H_0 \left[y + \frac{B}{A} \right] \right) \\
m_1 &= -1, \quad m_2 = \frac{B}{A}, \quad \beta_0 = -\frac{H_0}{A} \\
a^2 &= \frac{\rho A - \mu^* H_0^2}{\mu A}
\end{aligned}$$

The flow patterns for the unbounded domain and for the boundary value problem are given in Figures 3.9 and 3.10 respectively.

3.4.7 Example VII. (Flow with $xy - m_1 x^2 - m_2 x = \text{constant}$ as streamlines).

We take

$$y - m_1 x - m_2 = \frac{1}{x} \gamma(\psi)$$

and we get

$$\begin{aligned}
\gamma(\psi) &= a\psi + \psi_0 \\
\beta(\psi) &= \beta_0 \neq 0 \\
u &= \frac{1}{a} x, \quad v = \frac{1}{a} (2m_1 x + m_2 - y) \\
p &= \frac{\rho}{2a^2} (2m_2 y - x^2 - y^2) - \frac{2\mu^* m_1 \beta_0^2}{a^2} (m_1 x^2 + m_2 x - xy) + p_0
\end{aligned}$$

and the boundary conditions and constants are

$$\begin{aligned} u(0, y) &= 0, \quad v(0, y) = Ay + B, \quad A \neq 0 \\ \tilde{H}_{out} &= \left(-H_0x, H_0 \left[y + \frac{B}{A} \right] \right) \\ a &= -\frac{1}{A}, \quad m_2 = -\frac{B}{A}, \quad \beta_0 = \frac{H_0}{A} \end{aligned}$$

and m_1, p_0 are arbitrary constants. The streamlines for the unbounded domain and for the boundary value problem are shown in Figures 3.11 and 3.12 respectively.

3.4.8 Example VIII. (Flow with $x^3y = \text{constant}$ as streamlines).

We assume

$$x^3y = \gamma(\psi)$$

and have

$$\begin{aligned} \gamma(\psi) &= a\psi + \psi_0, \quad a \neq 0 \\ \beta(\psi) &= \sqrt{\frac{\rho}{\mu^*}} \\ u &= \frac{1}{a}x^3, \quad v = -\frac{3}{a}x^2y \\ H_1 &= \sqrt{\frac{\rho}{\mu^*}}u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}v \\ p &= p_0 - \frac{\rho}{2a^2} [x^6 + 9x^4y^2] + \frac{3\mu}{a} (x^2 - y^2) \end{aligned}$$

The boundary conditions and the constants are given by

$$\begin{aligned} u(0, y) &= 0, \quad v(0, y) = 0 \\ \tilde{H}_{out} &= (-H_0x^2, 2H_0xy) \end{aligned}$$

and $a \neq 0$ and p_0 are arbitrary constants. The flow patterns for the unbounded domain and for the boundary value problem are shown in Figures 3.13 and 3.14 respectively.

3.4.9 Example IX. (Flow with $\frac{y}{x} - m_1x^3 - \frac{m_2}{x^2} = \text{constant}$ as streamlines).

We let

$$y - m_1x^4 - \frac{m_2}{x} = x\gamma(\psi)$$

and obtain

$$\begin{aligned}
\gamma(\psi) &= -\frac{1}{6\mu} \left[\rho - \mu^* \beta_0^2 \right] \psi + \psi_0 \\
\beta(\psi) &= \beta_0, \quad \mu^* \beta_0^2 \neq \rho \\
u &= -\frac{6\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{x}, \quad v = \frac{6\mu}{[\rho - \mu^* \beta_0^2]} \left[\frac{2m_2}{x^3} - 3m_1 x^2 - \frac{y}{x^2} \right] \\
H_1 &= \beta_0 u, \quad H_2 = \beta_0 v \\
p &= \frac{6\mu^2}{[\rho - \mu^* \beta_0^2]^2} \left[\mu^* \beta_0^2 \left(12m_2 \frac{y}{x^5} - \frac{1}{x^2} - \frac{3y^2}{x^4} + 36m_1 m_2 \frac{1}{x} \right. \right. \\
&\quad \left. \left. - 27m_1^2 x^4 - \frac{12m_2^2}{x^6} + 42m_1 y \right) - 2\rho \left(\frac{1}{x^2} + 30m_1 y \right) \right] + p_0
\end{aligned}$$

The boundary conditions and the constants are given by

$$\begin{aligned}
u(x_0, y) &= Ax_0, \quad v(x_0, y) = Ay + B, \quad A \neq 0 \\
\tilde{H}_{out} &= H_0 \left(-x + 2x_0, y + \frac{B}{A} \right) \\
m_2 &= \frac{3}{2} m_1 x_0^5 - \frac{B}{2A} x_0, \quad \beta_0 = \frac{H_0}{A} \\
\frac{6\mu}{[\rho - \mu^* \beta_0^2]} &= -Ax_0^2
\end{aligned}$$

and m_1, p_0 are arbitrary constants. The streamlines for the unbounded domain and for the boundary value problem are shown in Figures 3.15 and 3.16 respectively.

3.4.10 Example X. (Flow with $y - x^3 - x^2 = \text{constant}$ as streamlines).

We take

$$y - x^3 - x^2 = \gamma(\psi)$$

and we get

$$\begin{aligned}
\gamma(\psi) &= a\psi + \psi_0, \quad a \neq 0 \\
\beta(\psi) &= \sqrt{\frac{\rho}{\mu^*}} \\
u &= \frac{1}{a}, \quad v = \frac{1}{a} (3x^2 + 2x) \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\
p &= p_0 - \frac{\rho}{a^2} \left(\frac{9}{2} x^4 + 6x^3 + 2x^2 \right) + \frac{6\mu}{a} y
\end{aligned}$$

The boundary conditions and the constants are

$$\begin{aligned} u(0, y) &= A \neq 0, \quad v(0, y) = 0 \\ \tilde{H}_{out} &= \left(\sqrt{\frac{\rho}{\mu^*}} A, x \right) \\ a &= \frac{1}{A} \end{aligned}$$

and p_0 is an arbitrary constant. The flow patterns for the unbounded domain and for the boundary value problem are shown in Figures 3.17 and 3.18 respectively.

3.4.11 Example XI. (Flow with $\sqrt{xy} = \text{constant}$ as streamlines).

We assume that

$$\sqrt{xy} = \gamma(\psi)$$

and we obtain

$$\begin{aligned} \gamma(\psi) &= \sqrt{\bar{\psi}} \\ \beta(\psi) &= \sqrt{\frac{\rho}{\mu^*}} \\ u &= 2xy, \quad v = -y^2 \\ H_1 &= \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\ p &= p_0 - 2\mu y - \frac{1}{2}\rho (4x^2y^2 + y^4) \end{aligned}$$

where p_0 is an arbitrary constant and the boundary conditions are given by

$$\begin{aligned} u(x, 0) &= 0, \quad v(x, 0) = 0 \\ \tilde{H}_{out} &= (H_0 y, 0) \end{aligned}$$

The streamline patterns for the unbounded domain and for the boundary value problem are shown in Figure 3.19 and 3.20 respectively.

3.4.12 Example XII. (Flow with $y - x^3 = \text{constant}$ as streamlines).

We take

$$y - x^3 = \gamma(\psi)$$

and we have

$$\gamma(\psi) = a\psi + \psi_0, \quad a \neq 0$$

$$\beta(\psi) = \sqrt{\frac{\rho}{\mu^*}}$$

$$u = \frac{1}{a}, \quad v = \frac{3}{a}x^2$$

$$H_1 = \sqrt{\frac{\rho}{\mu^*}}u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}}v$$

$$p = p_0 - \frac{9\rho}{2a^2}x^4 + \frac{6\mu}{a}y$$

The boundary conditions and the arbitrary constants are

$$u(0, y) = u_0 \neq 0, \quad v(0, y) = 0$$

$$\tilde{H}_{out} = \left(\sqrt{\frac{\rho}{\mu^*}}u_0, x \right)$$

$$a = \frac{1}{u_0}$$

and p_0 is an arbitrary constant. The streamlines for the unbounded domain and for the boundary value problem are shown in Figures 3.21 and 3.22 respectively.

$$Y - M_1 \cdot X^2 - M_2 \cdot X = C, M_1 = 1, M_2 = 1$$

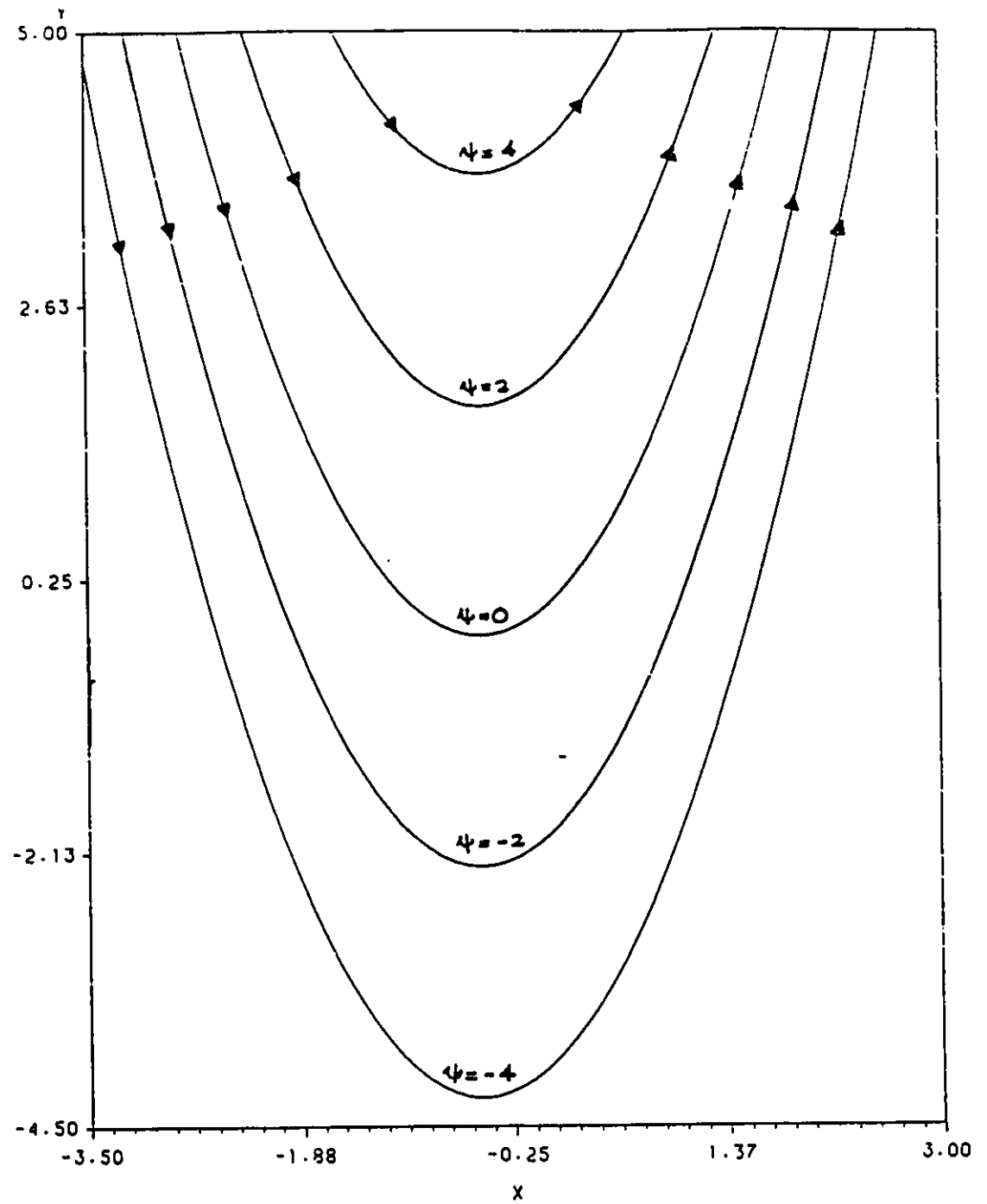


FIGURE 3.1: Streamline pattern for $y - m_1 x^2 - m_2 x = \text{constant}$ (unbounded domain).

$$Y - M_1 \cdot X^2 - M_2 \cdot X = C, M_1 = 1, M_2 = 1$$

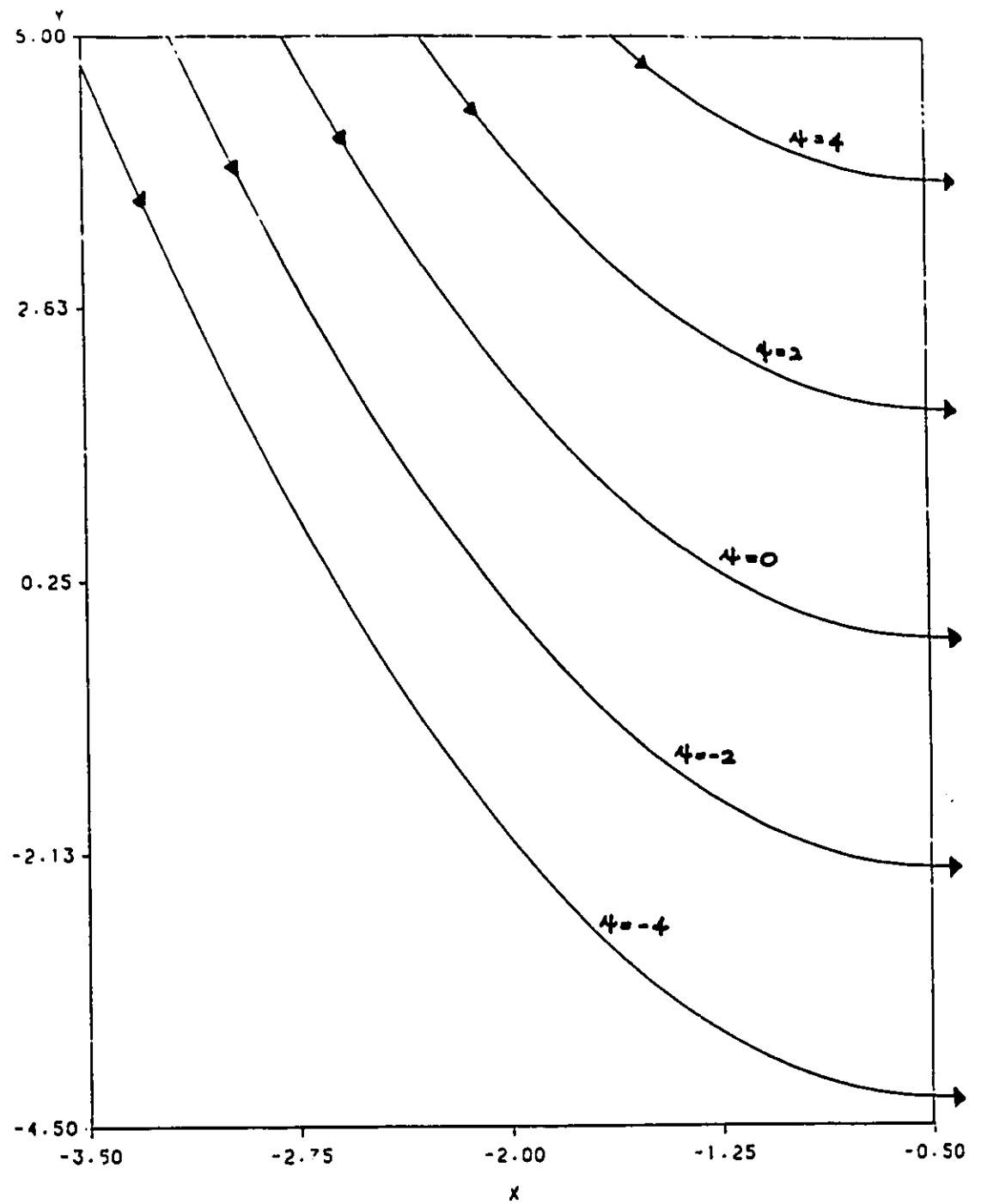


FIGURE 3.2: Streamline pattern for $y - m_1 x^2 - m_2 x = \text{constant}$ for boundary value problem.

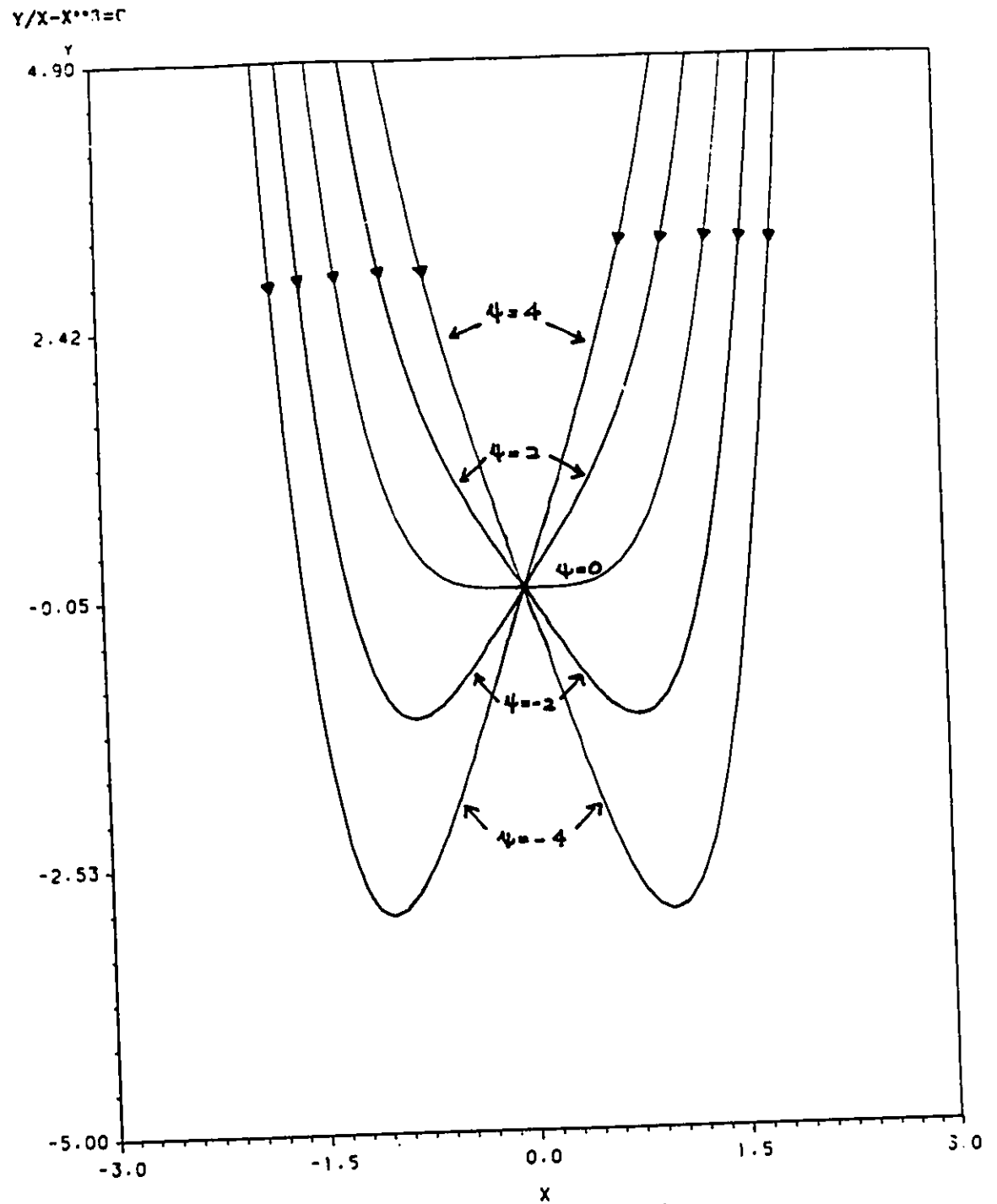


FIGURE 3.3: Streamline pattern for $\frac{y - ax^4}{x} = \text{constant}$
(unbounded domain).

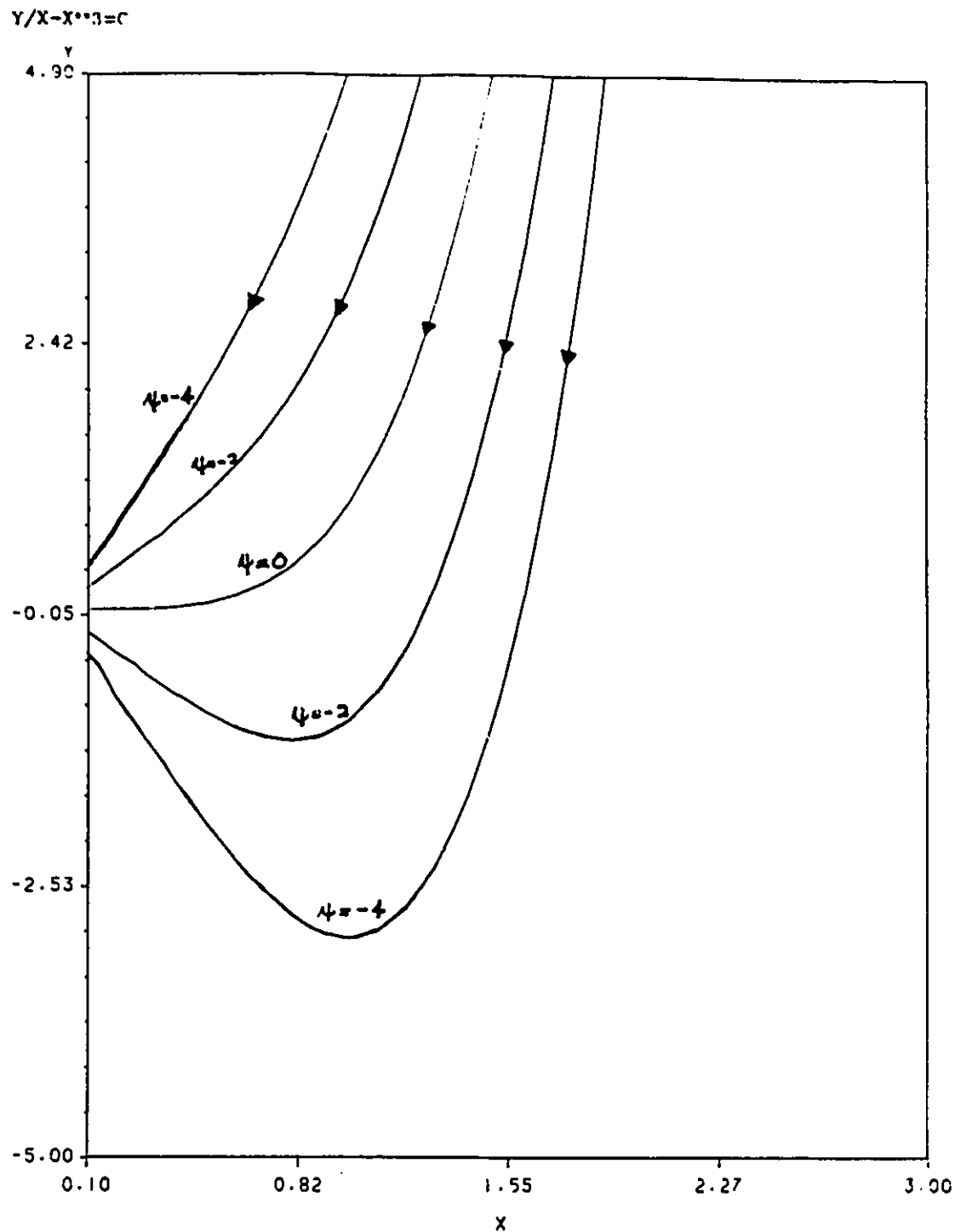


FIGURE 3.4: Streamline pattern for $\frac{y - az^4}{z} = \text{constant}$ for boundary value problem.

$$Y \cdot X^2 = C, M1=1, M2=1$$

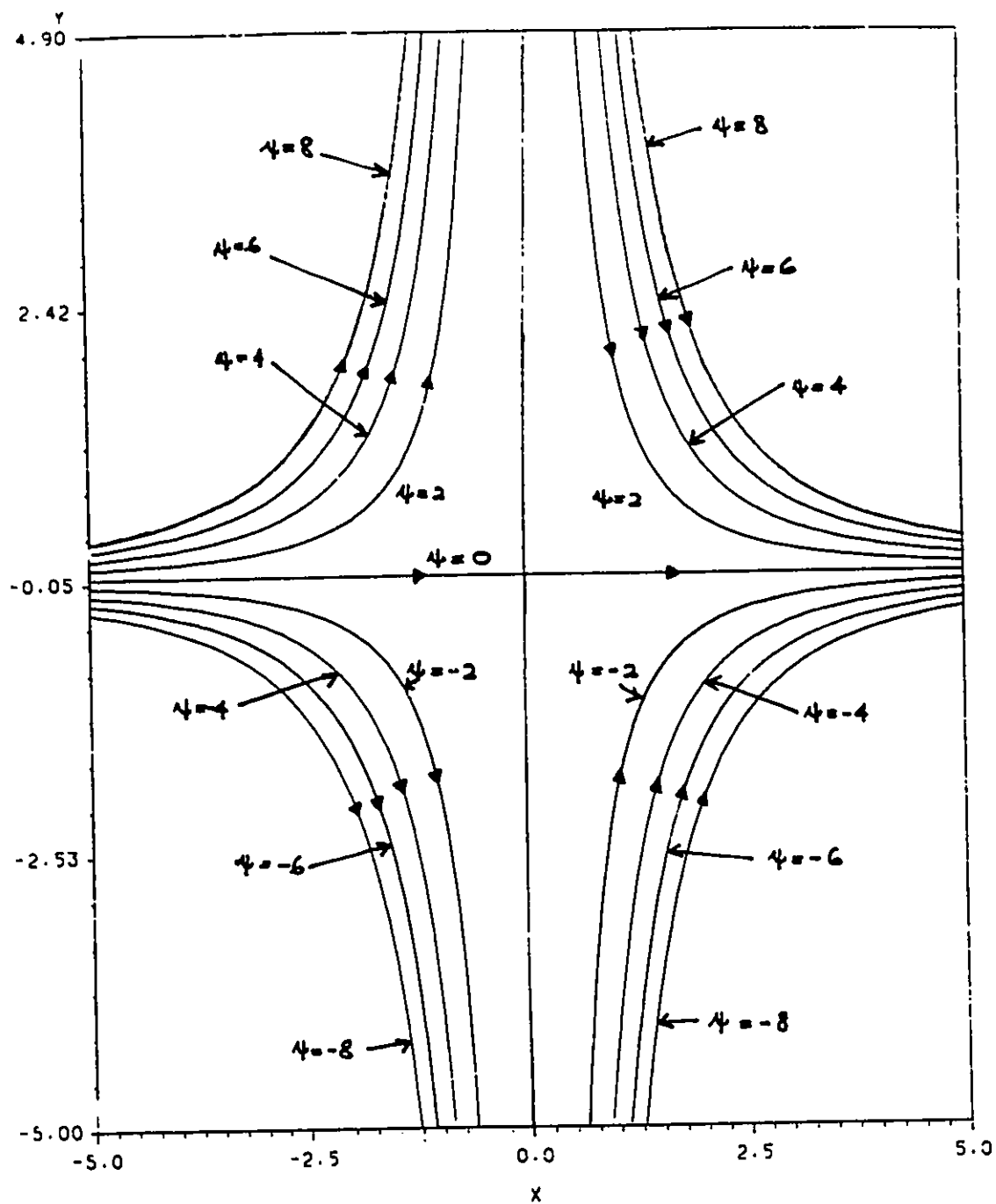


FIGURE 3.5: Streamline pattern for $x^2 y = \text{constant}$ (unbounded domain).

$$Y \cdot X^{**2} = C, M1=1, M2=1$$

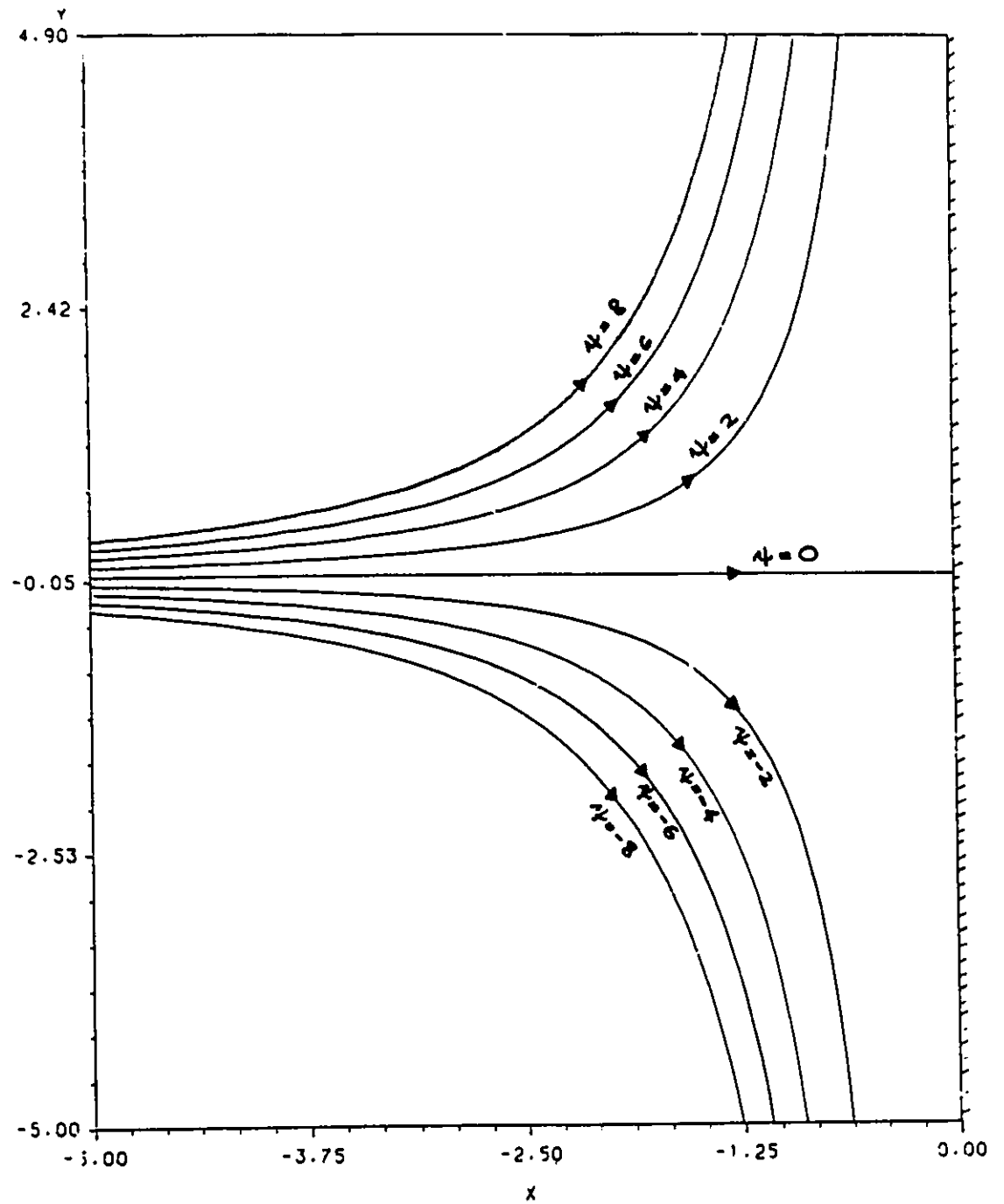


FIGURE 3.6: Streamline pattern for $x^2 y = \text{constant}$ for boundary value problem.

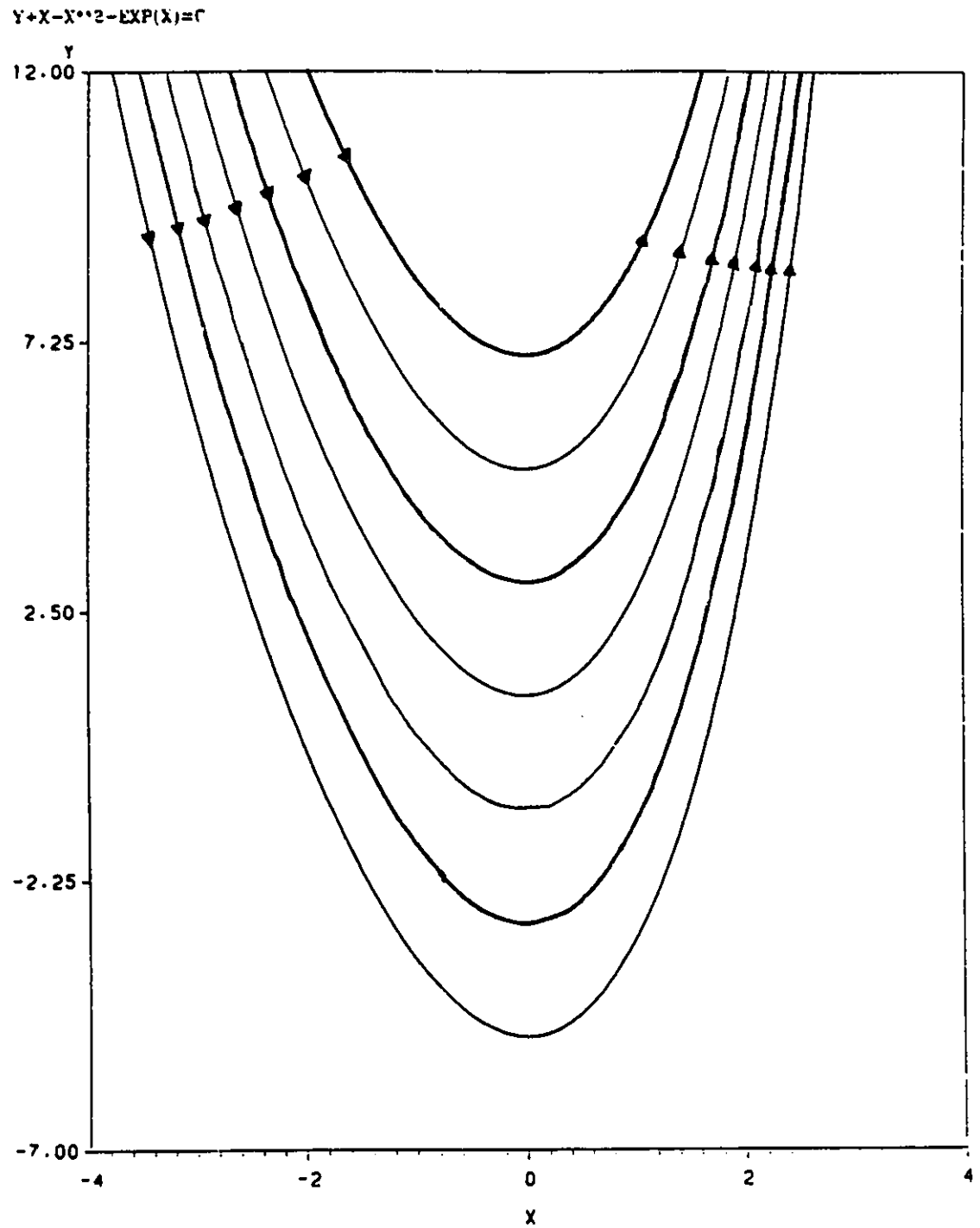


FIGURE 3.7: Streamline pattern for $y - e^{mx} - m_1x^2 - m_2x = \text{constant}$ (unbounded domain).

$$Y+X-X^2-EXP(Y)=C$$

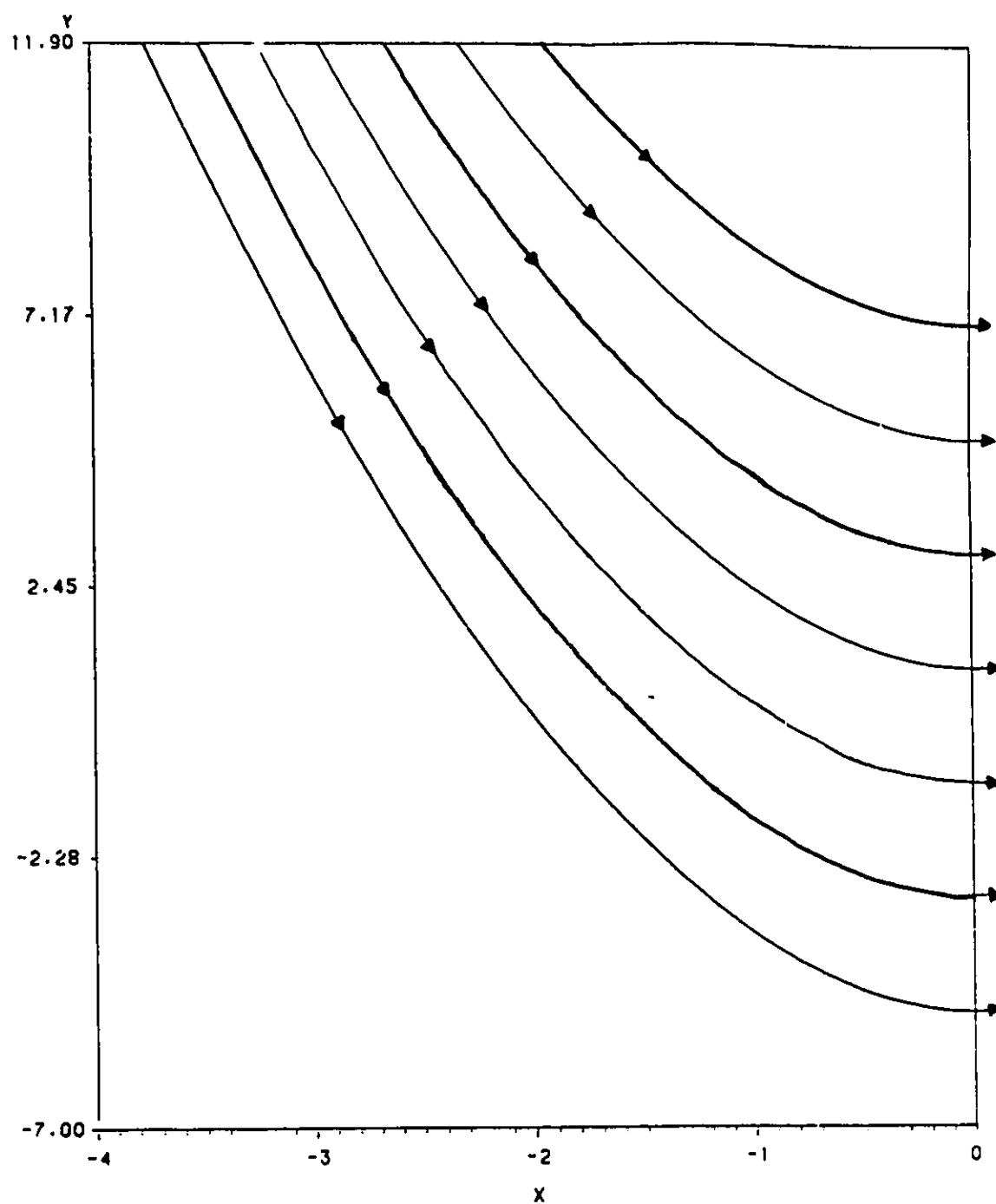


FIGURE 3.8: Streamline pattern for $y - e^{mx} - m_1x^2 - m_2x = \text{constant}$ for boundary value problem.

$$Y(1 - \exp(Y)) - \exp(Y) = C, M1=1, M2=1$$

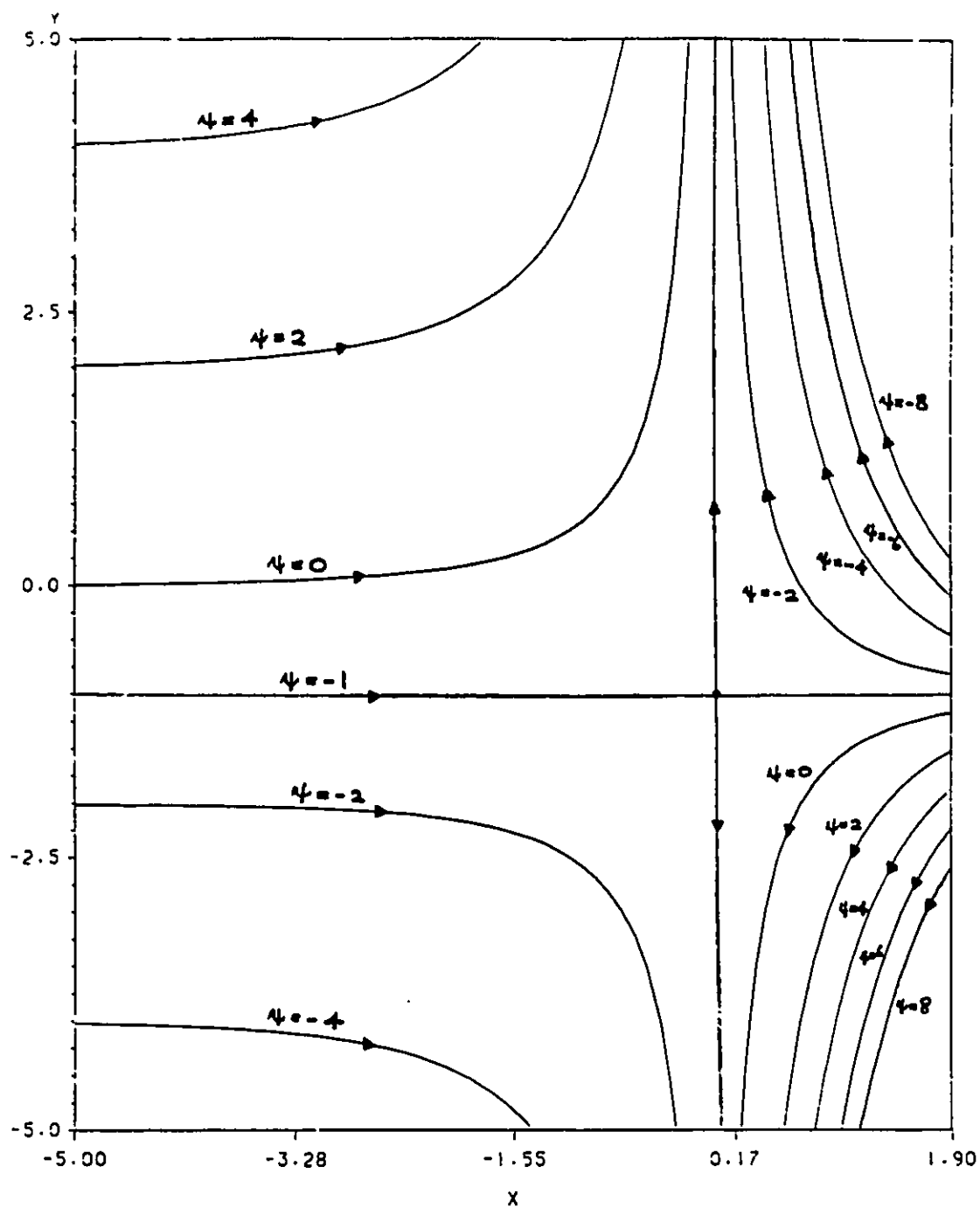


FIGURE 3.9: Streamline pattern for $y(1 + m_1 e^{ay}) - m_2 e^{ay} = \text{constant}$ (unbounded domain).

$$y \cdot (1 - \exp(\gamma)) - \exp(\gamma) = c$$

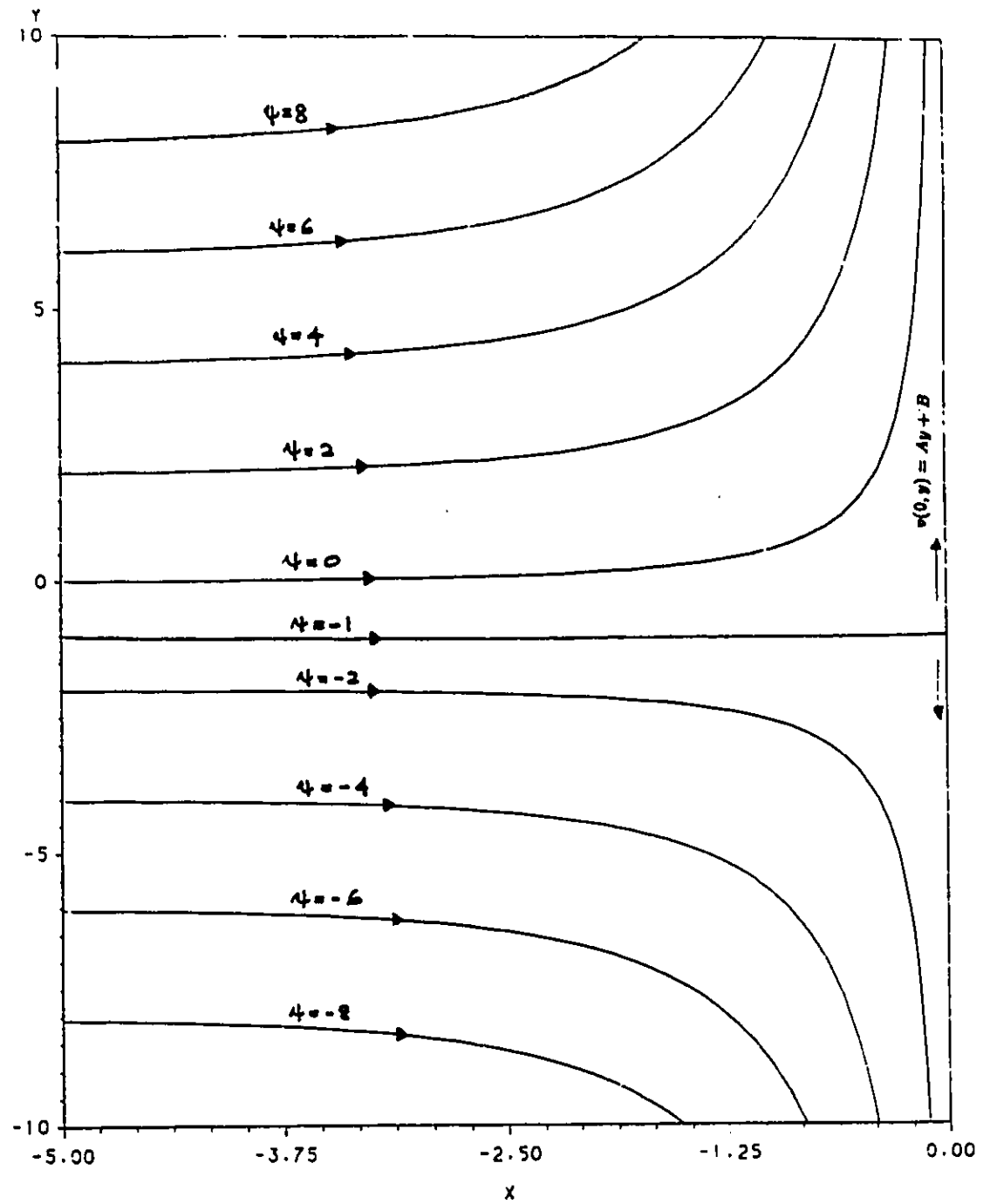


FIGURE 3.10: Streamline pattern for $y(1 + m_1 e^{ay}) - m_2 e^{ay} = \text{constant}$ for boundary value problem.

$$Y^*X - X^{**}2 - X = C$$

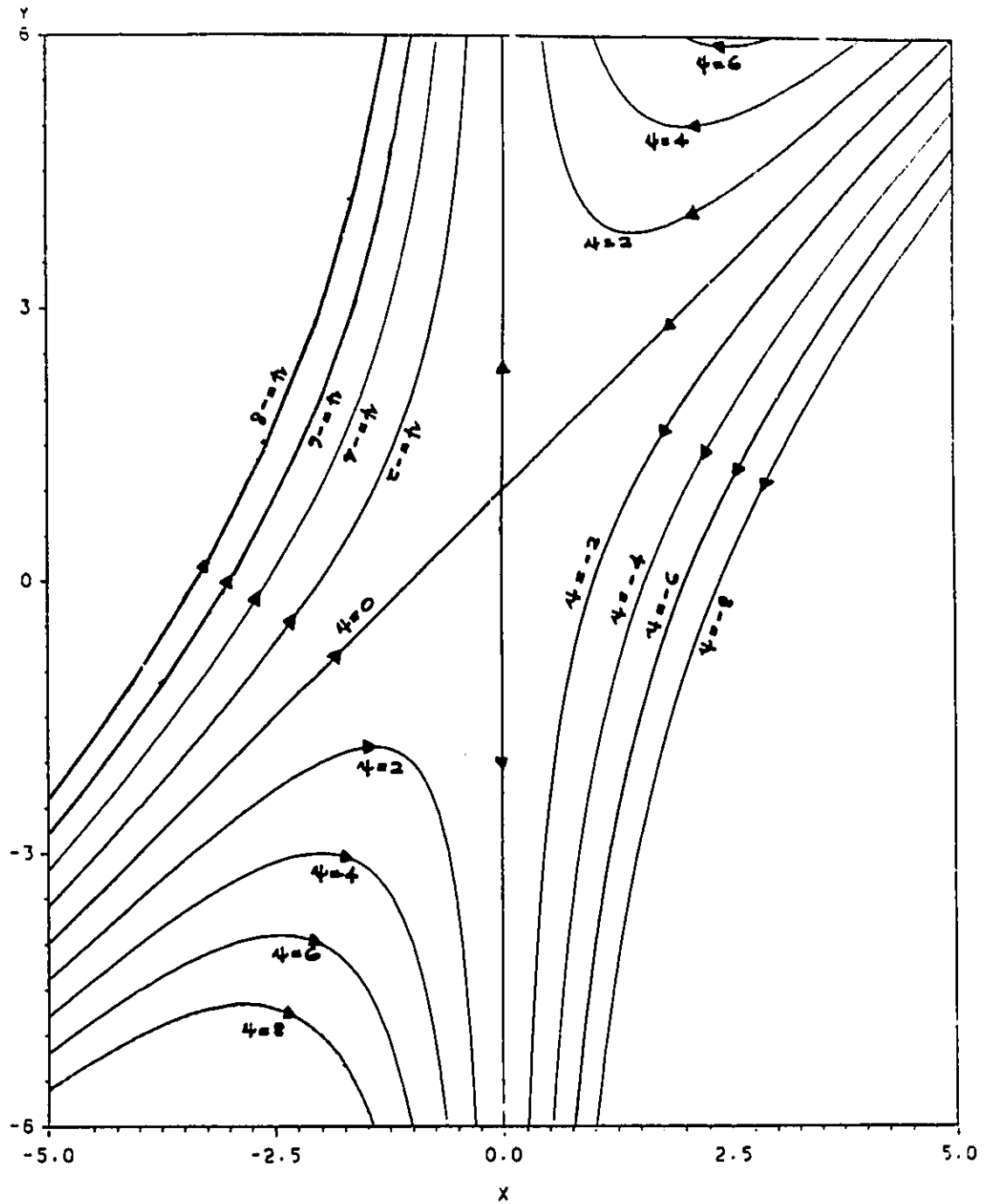


FIGURE 3.11: Streamline pattern for $xy - m_1x^2 - m_2x = \text{constant}$ (unbounded domain).

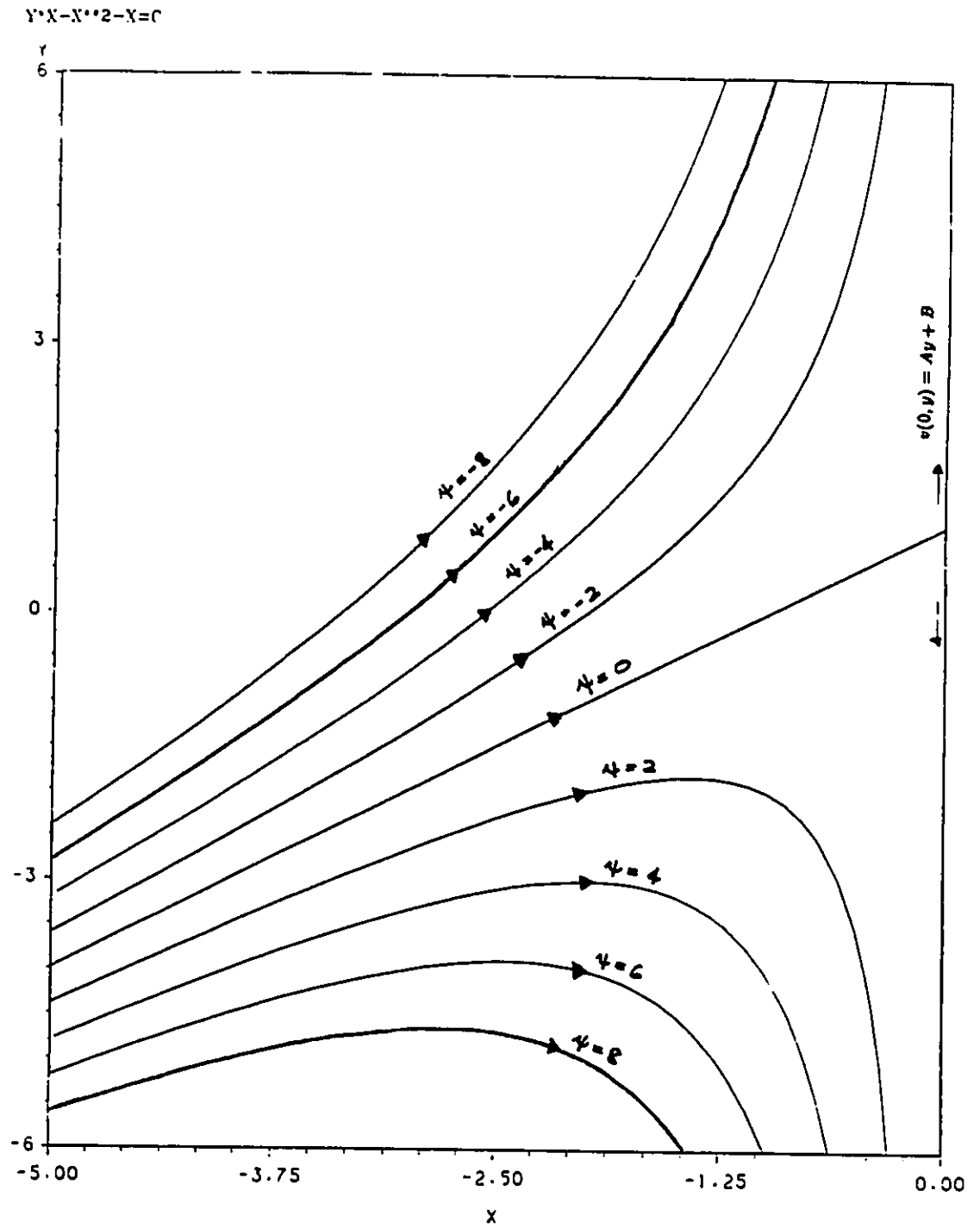


FIGURE 3.12: Streamline pattern for $xy - m_1x^2 - m_2x = \text{constant}$ for boundary value problem.

$$Y^*Y^{**3}=C, M1=1, M2=1$$

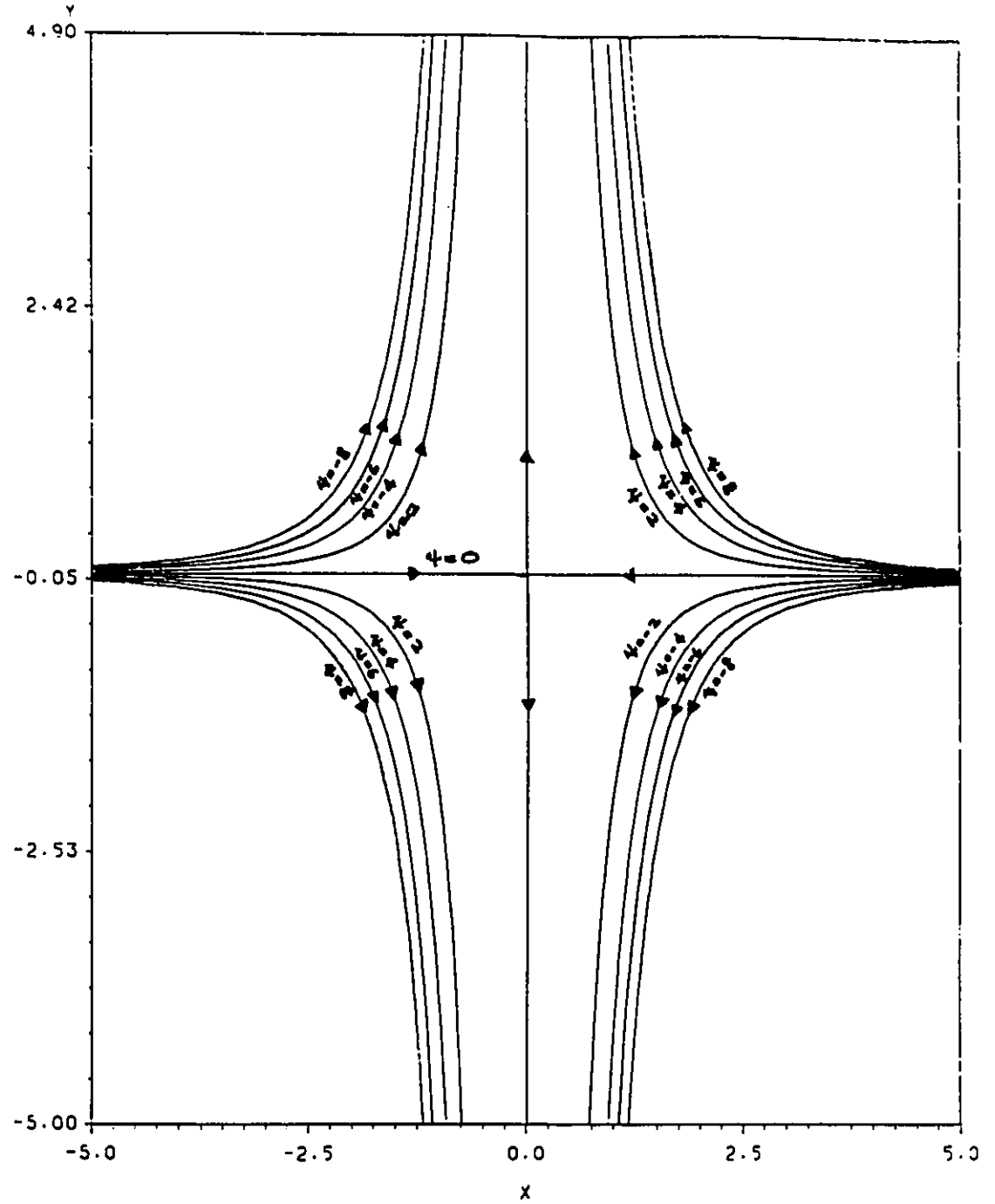


FIGURE 3.13: Streamline pattern for $x^3y = \text{constant}$
(unbounded domain).

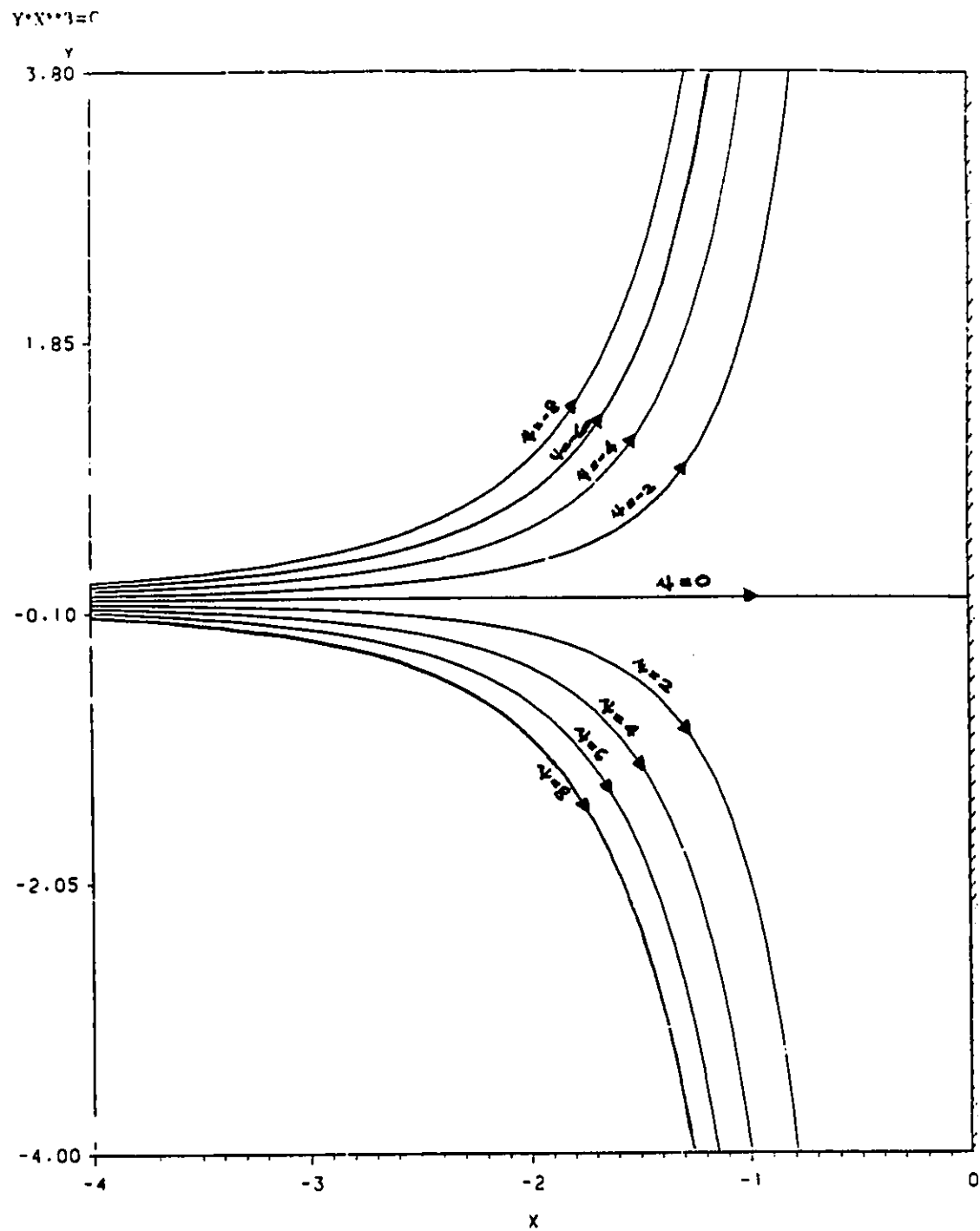


FIGURE 3.14: Streamline pattern for $x^{1/3}y = \text{constant}$ for boundary value problem.

$$Y/X - X^{**3} - 1/(X^{**2}) = C$$

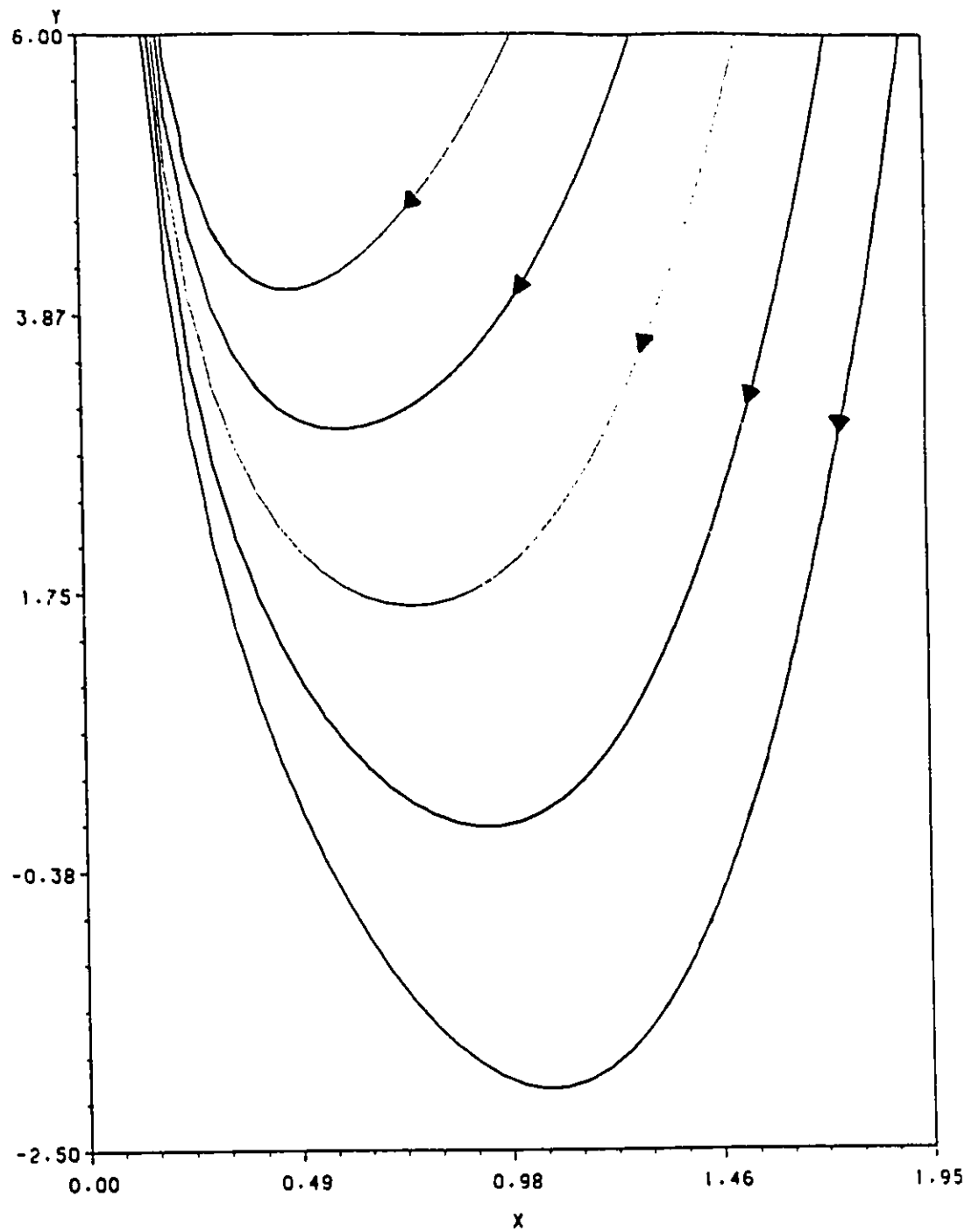


FIGURE 3.15: Streamline pattern for $\frac{y}{x} - m_1 x^3 - \frac{m_2}{x^2} = \text{constant}$ (unbounded domain).

$$Y/X - X^{**3} - 1/(X^{**2}) = C$$

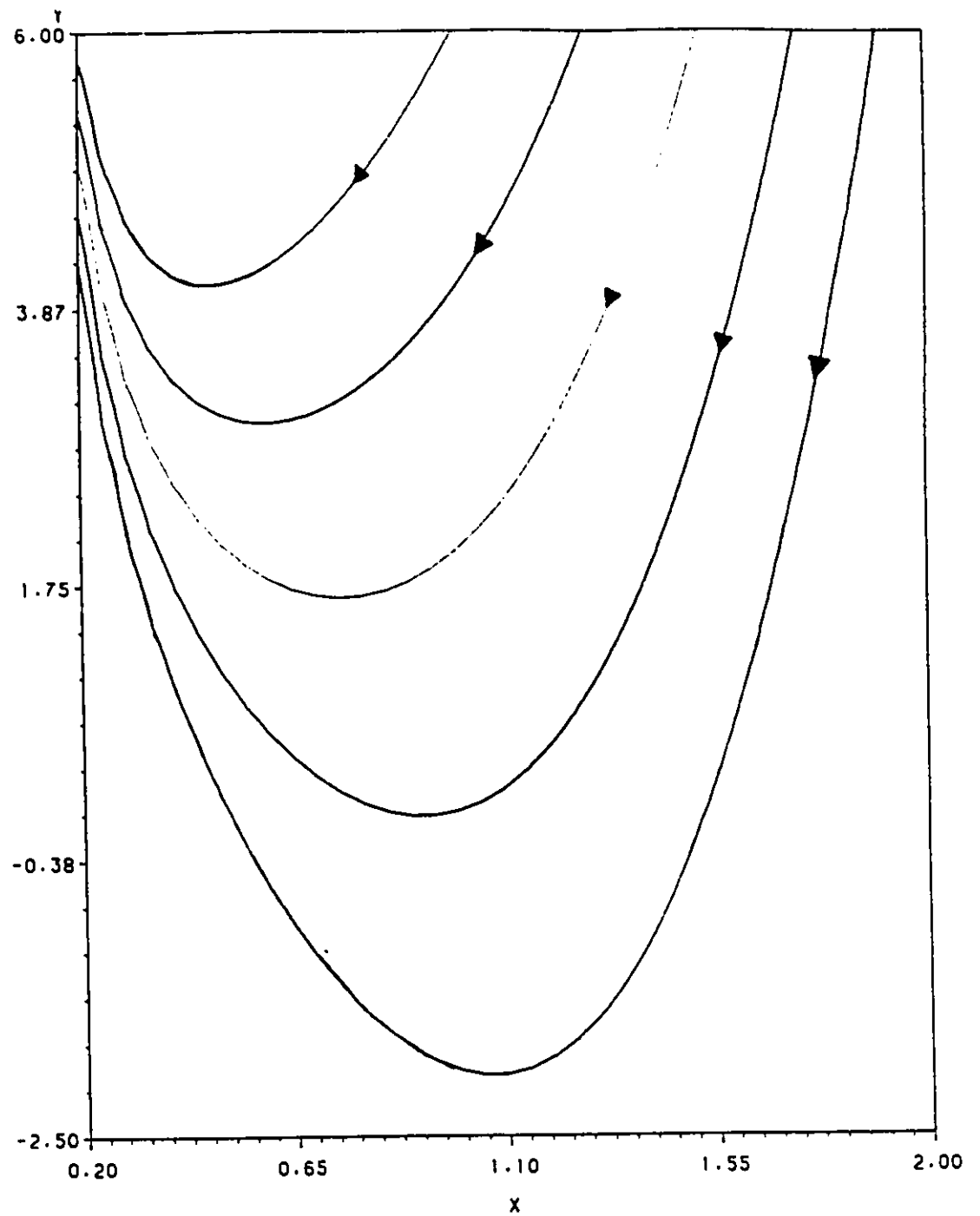


FIGURE 3.16: Streamline pattern, for $\frac{y}{x} - m_1 x^3 - \frac{m_2}{x^2} = \text{constant}$ for boundary-value problem.

$$Y - M1 \cdot X^{**3} - M2 \cdot X^{**2} = C, \quad M1=1, M2=1$$

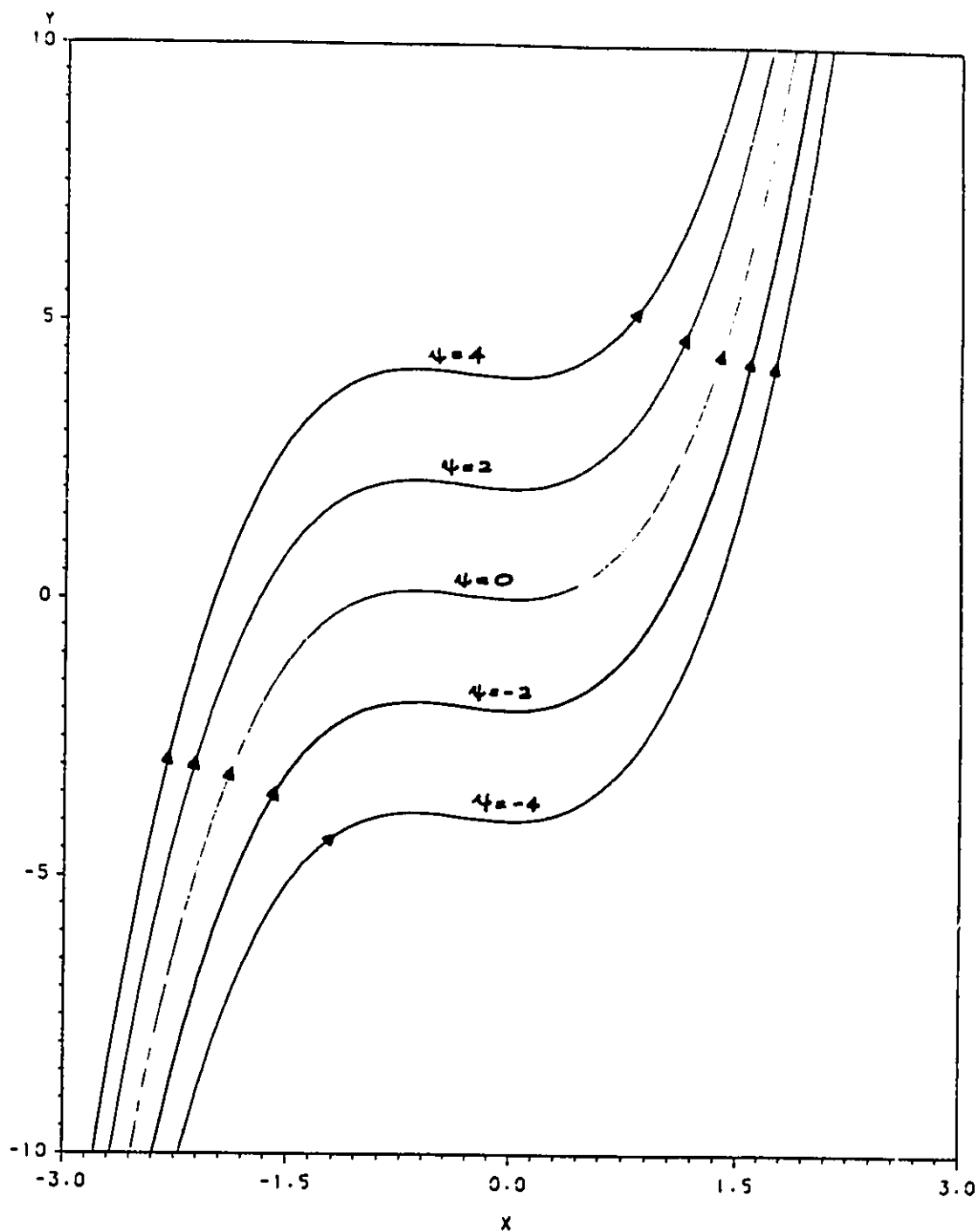


FIGURE 3.17: Streamline pattern for $y - x^3 - x^2 = \text{constant}$ (unbounded domain).

$$Y - X^{**3} - X^{**2} = C$$

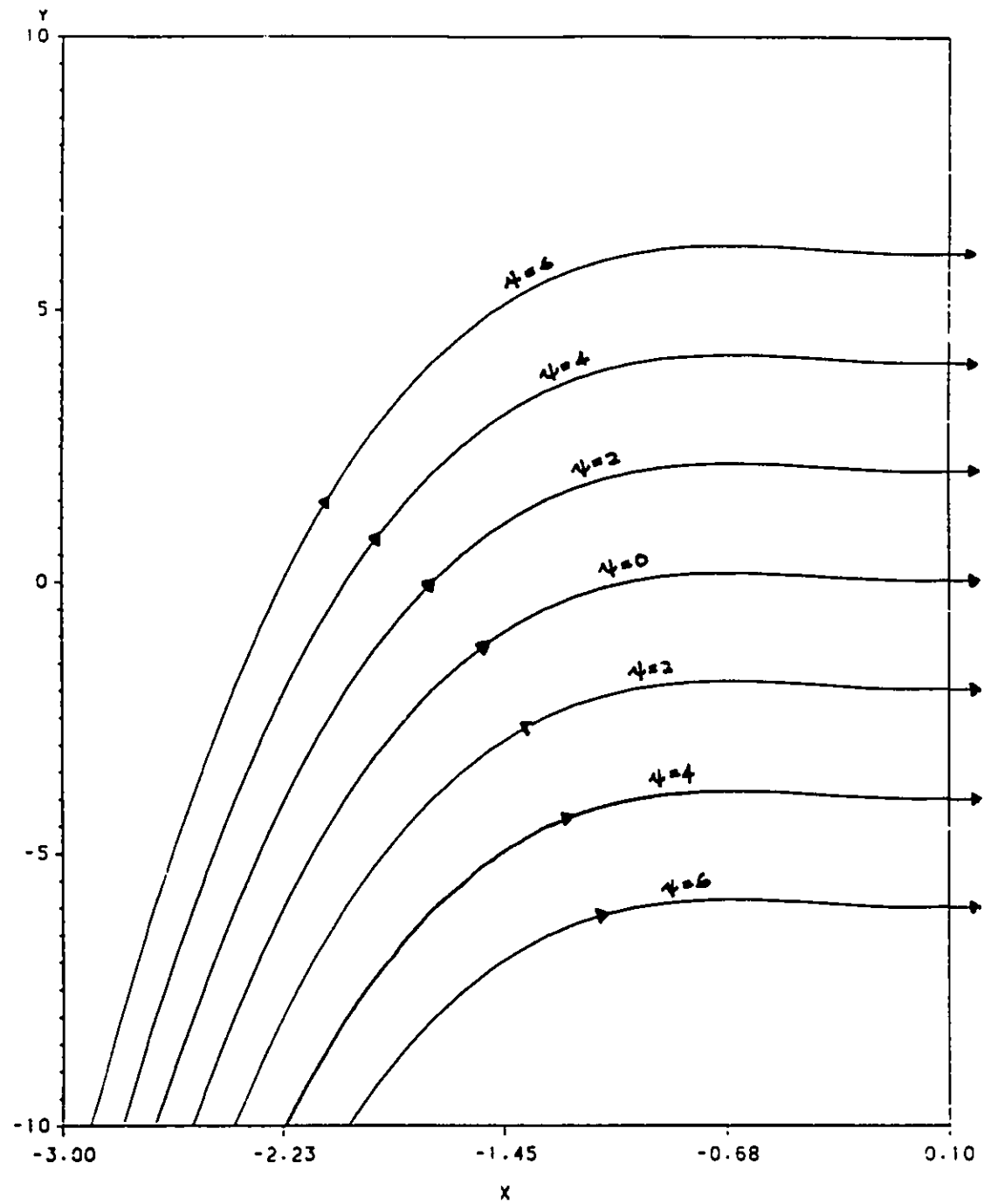


FIGURE 3.18: Streamline pattern for $y - x^3 - x^2 = \text{constant}$ for boundary value problem.

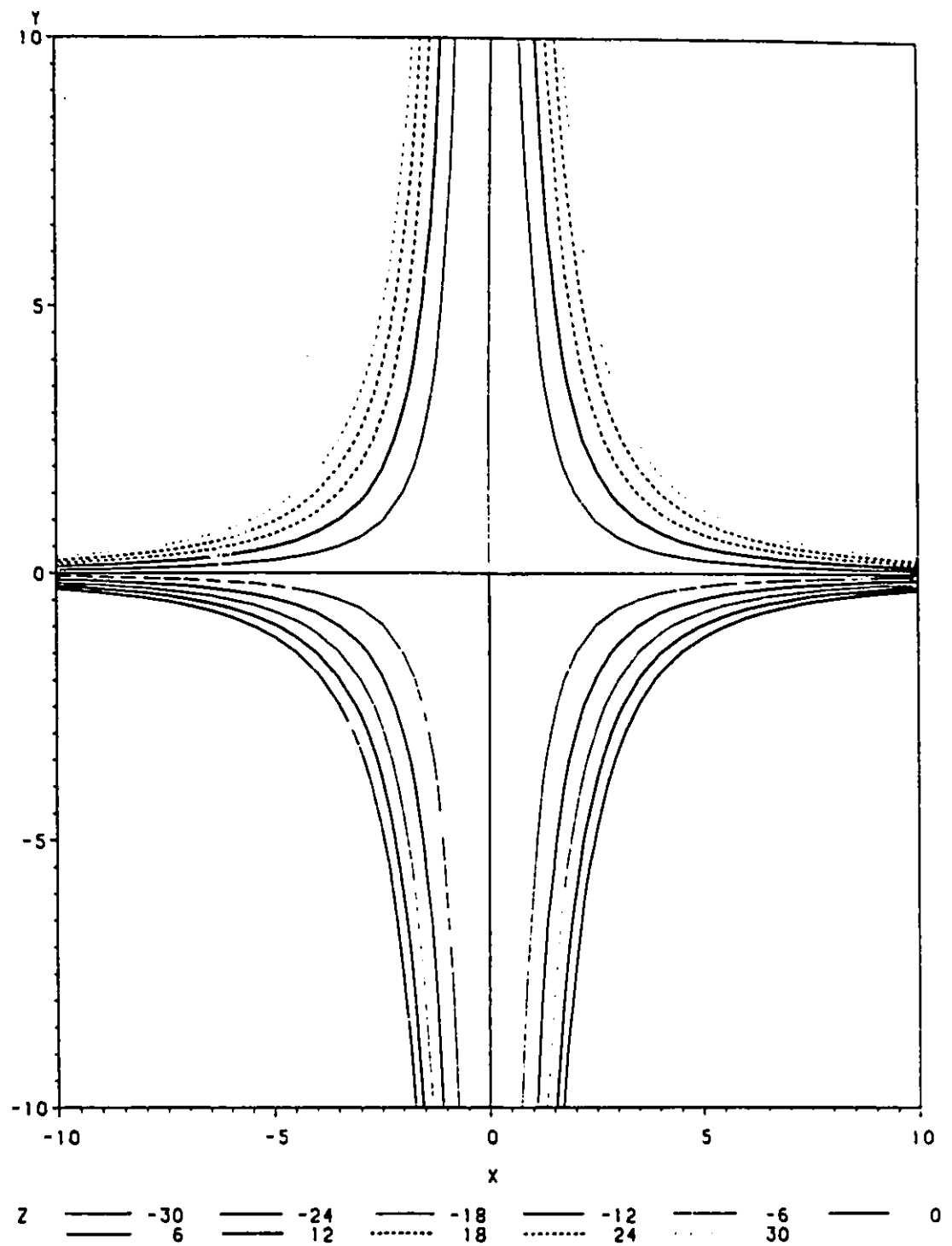


FIGURE 3.19: Streamline pattern for $\sqrt{xy} = \text{constant}$
(unbounded domain).

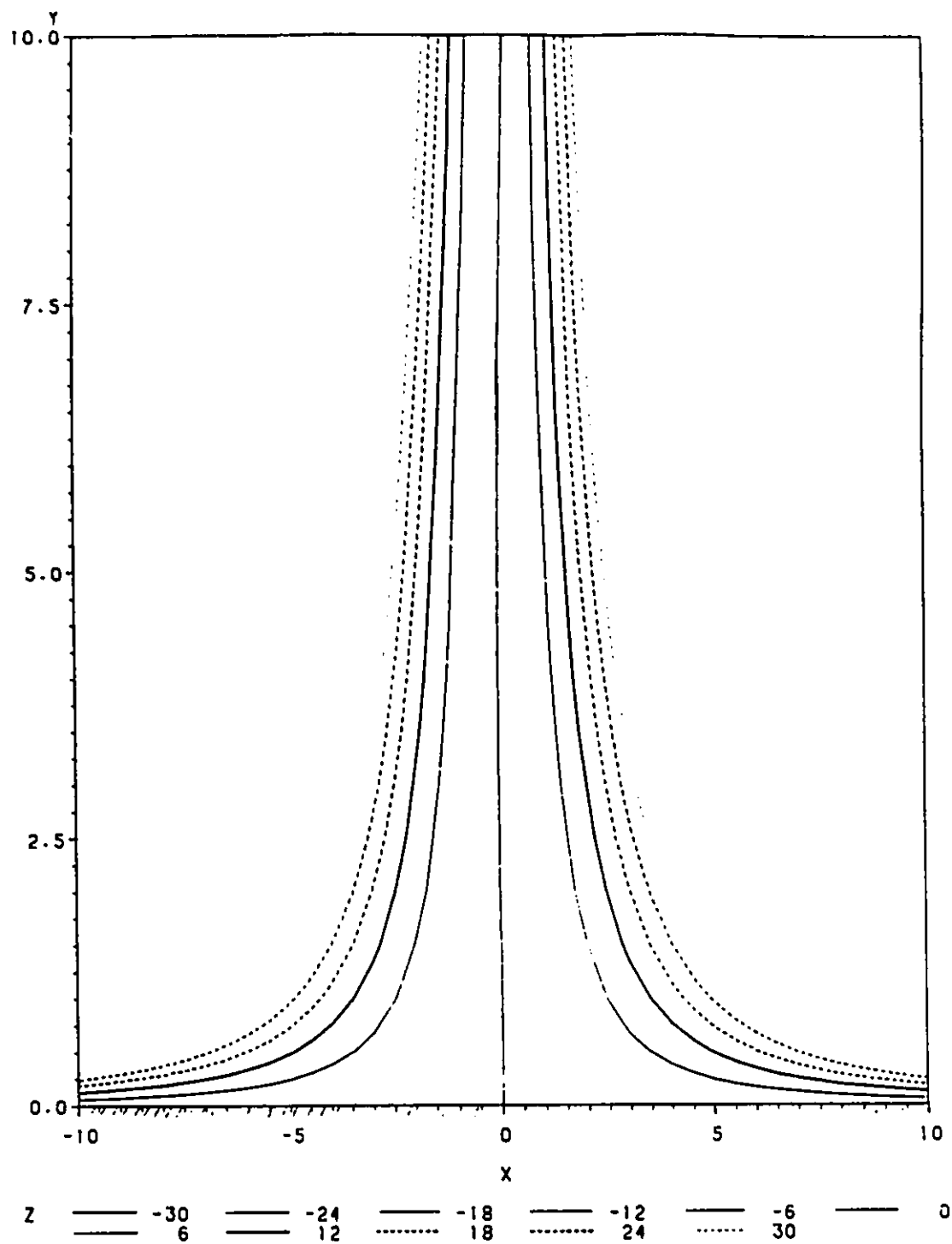


FIGURE 3.20: Streamline pattern for $\sqrt{xy} = \text{constant}$
for boundary-value problem.

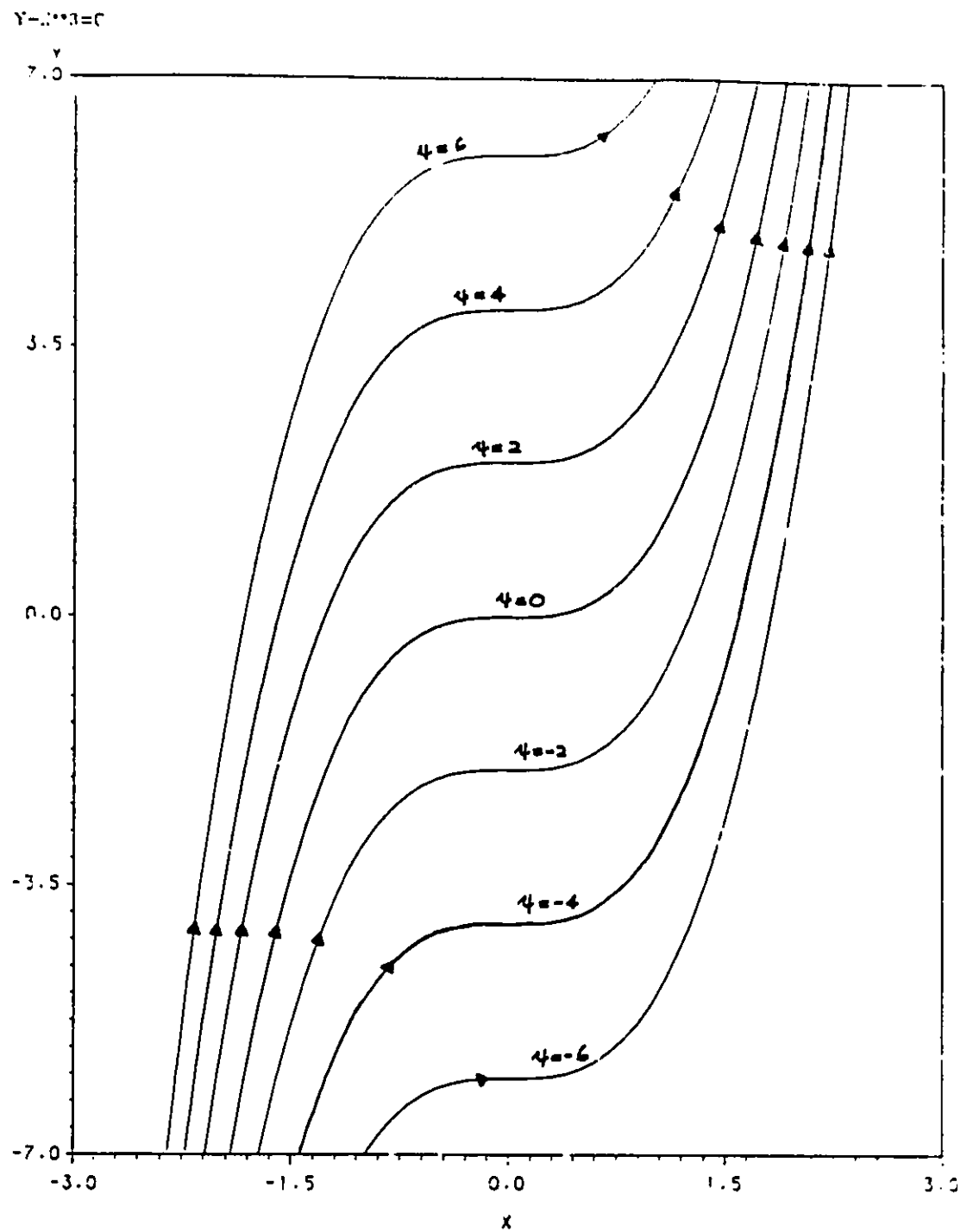


FIGURE 3.21: Streamline pattern for $y - x^3 = \text{constant}$
(unbounded domain).

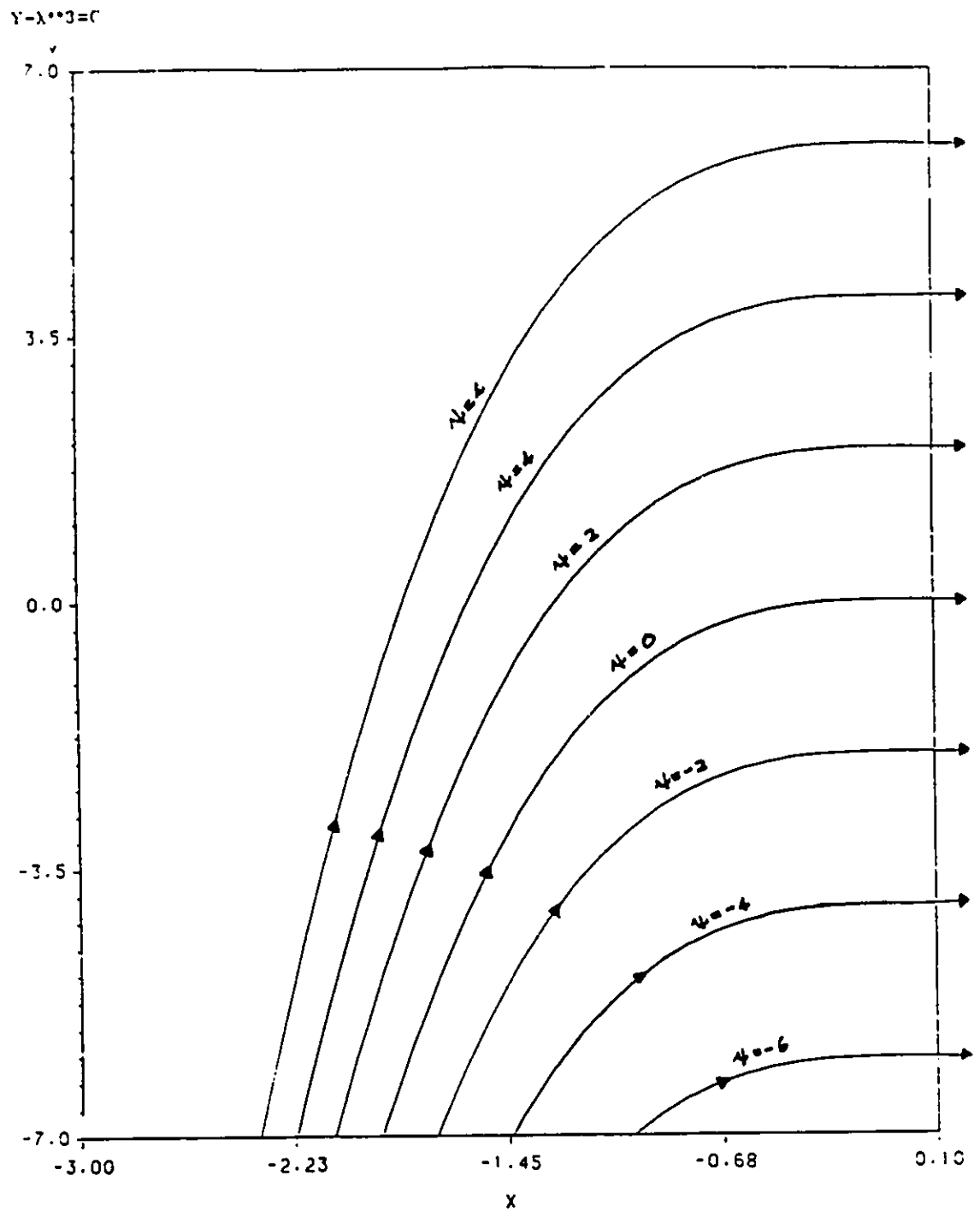


FIGURE 3.22: Streamline pattern for $y - z^3 = \text{constant}$
for boundary value problem.

CHAPTER 4

EXACT SOLUTIONS OF STEADY PLANE FLOWS USING (r, ψ) - COORDINATES

4.1 INTRODUCTION.

This chapter deals with flows for which polar representation of the streamline pattern is of the form $\frac{\theta - f(r)}{g(r)} = \text{constant}$. We study steady plane viscous incompressible MHD aligned and non-MHD fluid flows. We pose and answer the following two questions:

- (i) Given a family of plane curves $\frac{\theta - f(r)}{g(r)} = \text{constant}$, can fluid flow along these curves?
- (ii) Given a family of streamlines $\frac{\theta - f(r)}{g(r)} = \text{constant}$, what is the exact integral of the flow defined by the given streamline pattern?

To investigate our first question, we assume that fluid flows along the given family of curves $\frac{\theta - f(r)}{g(r)} = \text{constant}$. Since the streamfunction $\psi(r, \theta) = \text{constant}$ as well along these curves, it follows that there exists some function $\gamma(\psi)$ such that

$$\frac{\theta - f(r)}{g(r)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0$$

where $\gamma'(\psi)$ is the derivative of $\gamma(\psi)$.

We choose the (r, ψ) -coordinate system for our work in this chapter. Taking $v_1(r, \theta)$, $v_2(r, \theta)$ to be the components of velocity vector field in polar coordinates, we have

$$v_1 = \frac{1}{r} \frac{\partial \psi}{\partial \theta} = \frac{1}{r g(r) \gamma'(\psi)}, \quad v_2 = -\frac{\partial \psi}{\partial r} = \frac{1}{\gamma'(\psi)} \left[\frac{\theta g'(r)}{g^2(r)} + \left\{ \frac{f(r)}{g(r)} \right\}' \right]$$

4.2 FLOW EQUATIONS AND METHOD.

4.2.1. Infinitely Conducting MHD Aligned Flow.

For these flows, we have

$$\mathbf{H} = \beta \mathbf{V}, \quad \mathbf{H} \cdot \mathbf{k} = 0 \quad (4.1)$$

where $\beta(x, y)$ is an arbitrary function and \mathbf{k} is a unit vector in the z -direction. Using (4.1) in system of equations (2.10) to (2.18) with $\sigma \rightarrow \infty$ and $C = 0$, equations (2.13) to (2.15) are identically satisfied. Equation (2.10), written in polar coordinates, defines a streamfunction $\psi(r, \theta)$ such that $\psi(r, \theta) = \text{constant}$ are the streamlines. Letting the family of curves $r = \text{constant}$ be such that we have (r, ψ) -coordinate system in the physical plane and assuming that the fluid flows in the direction of increasing r along a streamline so that $J = W > 0$, the flow is governed by the following system

$$\begin{aligned} q &= \frac{\sqrt{E}}{J} \\ J \frac{\partial h}{\partial r} &= \mu \left[F \frac{\partial \omega}{\partial r} - E \frac{\partial \omega}{\partial \psi} \right] \\ J \frac{\partial h}{\partial \psi} &= \mu \left[-F \frac{\partial \omega}{\partial \psi} + G \frac{\partial \omega}{\partial r} \right] - [\rho \omega - \mu^* \beta(\psi) \Omega] J \\ \omega &= \frac{1}{J} \left[\frac{\partial}{\partial r} \left(\frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \right) \right] \\ \Omega &= \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} \\ \frac{\partial \beta}{\partial r} &= 0 \\ E &= \left[\frac{\partial}{\partial r} (r \cos \theta) \right]^2 + \left[\frac{\partial}{\partial r} (r \sin \theta) \right]^2 = 1 + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2 \\ F &= \frac{\partial}{\partial r} (r \cos \theta) \frac{\partial}{\partial \psi} (r \cos \theta) + \frac{\partial}{\partial r} (r \sin \theta) \frac{\partial}{\partial \psi} (r \sin \theta) = r^2 \frac{\partial \theta}{\partial r} \frac{\partial \theta}{\partial \psi} \end{aligned} \quad (4.2)$$

$$G = \left[\frac{\partial}{\partial \psi} (r \cos \theta) \right]^2 + \left[\frac{\partial}{\partial \psi} (r \sin \theta) \right]^2 = r^2 \left(\frac{\partial \theta}{\partial \psi} \right)^2$$

and

$$J = \frac{\partial}{\partial r} (r \cos \theta) \frac{\partial}{\partial \psi} (r \sin \theta) - \frac{\partial}{\partial \psi} (r \cos \theta) \frac{\partial}{\partial r} (r \sin \theta) = r \frac{\partial \theta}{\partial \psi}$$

of ten equations in ten unknown functions $E, F, G, J, \theta, \Omega, \omega, q, h$ and β of r, ψ .

Gauss equation is identically satisfied for our coordinate net.

Given a solution of system (4.2), the pressure function is known from

$$p = h - \frac{\rho E}{2J^2} \quad (4.3)$$

Furthermore, the flow in the physical plane is described by

$$\alpha = \int \left(\frac{\partial \alpha}{\partial r} dr + \frac{\partial \alpha}{\partial \psi} d\psi \right) \quad (4.4)$$

and

$$z = r e^{i\theta} = \int \frac{e^{i\theta}}{\sqrt{E}} \{ E dr + (F + iJ) d\psi \} \quad (4.5)$$

when

$$\frac{\partial \alpha}{\partial r} = \left[\frac{1}{1 + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2} \right] \left[2 \frac{\partial \theta}{\partial r} + r \frac{\partial^2 \theta}{\partial r^2} + r^2 \left(\frac{\partial \theta}{\partial r} \right)^3 \right] \quad (4.6)$$

and

$$\frac{\partial \alpha}{\partial \psi} = \left[\frac{1}{1 + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2} \right] \left[\frac{\partial \theta}{\partial \psi} + r \frac{\partial^2 \theta}{\partial r \partial \psi} + r^2 \left(\frac{\partial \theta}{\partial r} \right)^2 \left(\frac{\partial \theta}{\partial \psi} \right) \right] \quad (4.7)$$

We let $\frac{\theta - f(r)}{g(r)} = \text{constant}$ be the streamlines and we have

$$\frac{\theta - f(r)}{g(r)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (4.8)$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$.

Using (4.8), we find that E , F , G and J are given by

$$\begin{aligned} E &= 1 + r^2 [f'(r) + g'(r)\gamma(\psi)]^2, & G &= r^2 g^2(r) \gamma'^2(\psi) \\ F &= r^2 [f'(r) + g'(r)\gamma(\psi)] g(r) \gamma'(\psi), & J = W &= r g(r) \gamma'(\psi) \end{aligned} \quad (4.9)$$

Employing (4.9) and using the integrability condition $\frac{\partial^2 h}{\partial r \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial r}$, our flow is governed by:

$$\begin{aligned} & r g(r) \gamma'(\psi) \frac{\partial^2 \omega}{\partial r^2} - 2r [f'(r) + g'(r) \gamma(\psi)] \frac{\partial^2 \omega}{\partial r \partial \psi} + \left[\frac{1}{r g(r)} + \frac{r f'^2(r)}{g(r)} \right. \\ & + \left. \frac{2r f'(r) g'(r)}{g(r)} \gamma(\psi) + \frac{r g'^2(r)}{g(r)} \gamma^2(\psi) \right] \frac{1}{\gamma'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} + \left\{ -f'(r) - r f''(r) + 2r \frac{f'(r) g'(r)}{g(r)} \right. \\ & + \left[2r \frac{g'^2(r)}{g(r)} - g'(r) - r g''(r) \right] \gamma(\psi) - \left[\frac{1}{r g(r)} + \frac{r f'^2(\psi)}{g(r)} \right] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ & - \left. \frac{2r f'(r) g'(r)}{g(r)} \frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} - \frac{r g'^2(r)}{g(r)} \frac{\gamma^2(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \right\} \frac{\partial \omega}{\partial \psi} + \left[g(r) \gamma'(\psi) - \frac{\rho}{\mu} \right] \frac{\partial \omega}{\partial r} \\ & + \frac{\mu^*}{\mu} \beta(\psi) \frac{\partial \Omega}{\partial r} = 0 \end{aligned} \quad (4.10)$$

$$\begin{aligned} \omega &= \left[\frac{1}{r} \frac{f'(r)}{g(r)} + \frac{f''(r)}{g(r)} - \frac{2f'(r)g'(r)}{g^2(r)} \right] \frac{1}{\gamma'(\psi)} + \left[\frac{1}{r} \frac{g'(r)}{g(r)} + \frac{g''(r)}{g(r)} - \frac{2g'^2(r)}{g^2(r)} \right] \frac{\gamma(\psi)}{\gamma'(\psi)} \\ & + \left[\frac{1}{r^2 g^2(r)} + \frac{f'^2(r)}{g^2(r)} \right] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2f'(r)g'(r)}{g^2(r)} \frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^3(\psi)} + \frac{g'^2(r)}{g^2(r)} \frac{\gamma^2(\psi) \gamma''(\psi)}{\gamma'^3(\psi)} \end{aligned} \quad (4.11)$$

$$\Omega = \beta(\psi) \omega - \frac{1 + r^2 [f'(r) + g'(r) \gamma(\psi)]^2}{r^2 g^2(r) \gamma'^2(\psi)} \beta'(\psi) \quad (4.12)$$

of three equations in four unknown functions ω , Ω , $\gamma(\psi)$ and $\beta(\psi)$. Equation (4.10) is one equation in two unknown functions when ω and Ω are eliminated using (4.11) and (4.12). Summing up, we have the following theorem:

Theorem 4.1. *If a steady, plane, viscous, incompressible, electrically conducting fluid of infinite electrical conductivity flows along $\frac{\theta - f(r)}{g(r)} = \text{constant}$ in the presence of an MHD aligned field, then the known functions $f(r)$, $g(r)$ and the unknown functions $\beta(\psi)$, $\gamma(\psi)$ must satisfy equation (4.10) where ω and Ω are given by equations (4.11) and (4.12) respectively.*

4.2.2 Finitely Conducting Flow.

In this case, $\Omega = \Omega_0$ where Ω_0 is an arbitrary constant. Using (4.11) in (4.10) and (4.12), we get two coupled equations in unknown functions $\gamma(\psi)$ and $\beta(\psi)$ and, therefore, we have:

Theorem 4.2. *If a steady, plane, viscous, incompressible, electrically conducting fluid of finite electrical conductivity flows along $\frac{\theta - f(r)}{g(r)} = \text{constant}$, in the presence of an aligned magnetic field, then the known functions $f(r)$, $g(r)$ and the unknown functions $\beta(\psi)$, $\gamma(\psi)$ must satisfy*

$$\begin{aligned} & rg(r)\gamma'(\psi)\frac{\partial^2\omega}{\partial r^2} - 2r[f'(r) + g'(r)\gamma(\psi)]\frac{\partial^2\omega}{\partial r\partial\psi} + \left[\frac{1}{rg(r)} + \frac{rf'^2(r)}{g(r)}\right. \\ & + \left.\frac{2rf'(r)g'(r)}{g(r)}\gamma(\psi) + \frac{rg'^2(r)}{g(r)}\gamma^2(\psi)\right]\frac{1}{\gamma'(\psi)}\frac{\partial^2\omega}{\partial\psi^2} + \left\{-f'(r) - rf''(r) + 2r\frac{f'(r)g'(r)}{g(r)}\right. \\ & + \left[2r\frac{g'^2(r)}{g(r)} - g'(r) - rg''(r)\right]\gamma(\psi) - \left[\frac{1}{rg(r)} + \frac{rf'^2(\psi)}{g(r)}\right]\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ & \left. - \frac{2rf'(r)g'(r)}{g(r)}\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{rg'^2(r)}{g(r)}\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right\}\frac{\partial\omega}{\partial\psi} + \left[g(r)\gamma'(\psi) - \frac{\rho}{\mu}\right]\frac{\partial\omega}{\partial r} = 0 \end{aligned} \quad (4.13)$$

$$\beta(\psi)\omega - \frac{1 + r^2[f'(r) + g'(r)\gamma(\psi)]^2}{r^2g^2(r)\gamma'^2(\psi)}\beta'(\psi) = \Omega_0 \quad (4.14)$$

where ω is given by equation (4.11).

4.2.3 Non-MHD Flow.

In the case of non-MHD fluid flow, $\Omega = \beta = 0$ and equation (4.10) becomes one equation in one unknown function $\gamma(\psi)$ after (4.11) is used. We, therefore have:

Theorem 4.3. *If a steady, plane, viscous, incompressible fluid flows along $\frac{\theta - f(r)}{g(r)} = \text{constant}$, then the known functions $f(r)$, $g(r)$ and the unknown function $\gamma(\psi)$ must satisfy equation (4.10) with ω given by (4.11) and $\Omega = 0$.*

4.3 APPLICATIONS.

This section deals with various flows to illustrate the method.

4.3.1 Example I. (Flow with $\theta - m_1 r^3 - m_2 r^2 = \text{constant}$ as streamlines).

We assume that

$$\theta = m_1 r^3 + m_2 r^2 + \gamma(\psi); \quad \gamma'(\psi) \neq 0, \quad m_1 \neq 0 \quad (4.15)$$

where $\gamma(\psi)$ is an unknown function of ψ .

Comparing (4.15) with (4.8), we have

$$f(r) = m_1 r^3 + m_2 r^2, \quad g(r) = 1 \quad (4.16)$$

The streamline pattern for this flow is shown in Figure 4.1.

Infinitely Conducting Flow

Employing (4.11), (4.12) and (4.16) in (4.10), we get

$$\sum_{n=0}^{12} A_n(\psi) r^n = 0 \quad (4.17)$$

$$n \neq 1, 2$$

where

$$\begin{aligned} A_0(\psi) &= \frac{2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ &\quad + \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} \\ A_3(\psi) &= 9m_1 - 6m_1 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{9m_1}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\ A_4(\psi) &= 8m_2^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 32m_2^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ &\quad - \frac{8m_2^2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ A_5(\psi) &= 24m_1 m_2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 216m_1 m_2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ &\quad - \frac{36m_1 m_2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \end{aligned}$$

$$\begin{aligned}
A_6(\psi) &= 18m_1^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 72m_2^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 279m_1^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad - \frac{36m_1^2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_7(\psi) &= -360m_1m_2^2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
A_8(\psi) &= 16m_2^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 648m_1^2m_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
A_9(\psi) &= 96m_1m_2^3 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 378m_1^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
A_{10}(\psi) &= 216m_1^2m_2^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
A_{11}(\psi) &= 216m_1^3m_2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
A_{12}(\psi) &= 81m_1^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Equation (4.17) is a polynomial of degree twelve in r with coefficients as functions of ψ only. Since r, ψ are independent variables it follows that equation (4.17) can only hold true for all values of r if the coefficients of different power of r vanish simultaneously and we have:

$$A_n(\psi) = 0, \quad n = 0, 3, 4, \dots, 12$$

Using $m_1 \neq 0$ and $A_{12}(\psi) = 0$ in $A_9(\psi) = 0$, we get

$$\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \quad (4.18)$$

Substituting equation (4.18) in $A_3(\psi) = 0$, we have

$$\frac{1}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} - 1 = 0 \quad (4.19)$$

Employing (4.18) and (4.19) in $A_0(\psi) = 0$, we obtain

$$3 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{\mu^*}{\mu} \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} = 0 \quad (4.20)$$

Upon substitution of (4.18) to (4.20), $A_6(\psi) = 0$ yields

$$\gamma''(\psi) = 0$$

which upon integration with respect to ψ gives

$$\gamma(\psi) = a_1\psi + \psi_0 \quad (4.21)$$

where $a_1 \neq 0$ and ψ_0 are arbitrary constants. Using (4.21) in (4.20) and integrating the resulting equation, we get $\beta(\psi) = \beta_0$ where $\beta_0 \neq 0$ is an arbitrary constant. Employing $\beta(\psi) = \beta_0$ and (4.21) in (4.19), we get $a_1 = \frac{1}{\mu} [\rho - \mu^* \beta_0^2]$. Thus, the unknown functions $\gamma(\psi)$ and $\beta(\psi)$ are given by

$$\gamma(\psi) = \frac{1}{\mu} (\rho - \mu^* \beta_0^2) \psi + \psi_0, \quad \beta(\psi) = \beta_0 \quad (4.22)$$

where ψ_0 and $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ are arbitrary constants. Using (4.15), (4.22) and $v_1 = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$, $v_2 = -\frac{\partial \psi}{\partial r}$, we find that the solutions are given by

$$\begin{aligned} v_1 &= \frac{\mu}{\rho - \mu^* \beta_0^2} \frac{1}{r}, \\ v_2 &= \frac{\mu}{\rho - \mu^* \beta_0^2} [3m_1 r^2 + 2m_2 r] \\ H_1 &= \frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]} \left[\frac{\cos \theta}{r} - (3m_1 r^2 + 2m_2 r) \sin \theta \right] \\ H_2 &= \frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]} \left[\frac{\sin \theta}{r} + (3m_1 r^2 + 2m_2 r) \cos \theta \right] \\ p &= \frac{\mu^2}{(\rho - \mu^* \beta_0^2)^2} \left\{ \frac{9m_1^2}{4} (\rho - 3\mu^* \beta_0^2) r^4 - \frac{\rho}{2r^2} + 2m_2^2 (\rho - 2\mu^* \beta_0^2) r^2 \right. \\ &\quad \left. + 2m_1 m_2 (2\rho - 5\mu^* \beta_0^2) r^3 - 4m_2 (\rho - 2\mu^* \beta_0^2) [\theta - \psi_0] \right\} + p_0 \\ \omega &= \frac{\mu}{[\rho - \mu^* \beta_0^2]} [9m_1 r + 2m_2] \\ \Omega &= \beta_0 \omega \end{aligned} \quad (4.23)$$

where p_0 is an arbitrary constant. Since the pressure p must be a single-valued function, we must take $m_2 = 0$.

Finitely Conducting Flow

Using (4.11) and (4.16) in (4.13) and (4.14), we get

$$\sum_{n=0}^{12} B_n(\psi) r^n = 0 \quad (4.24)$$

$$n \neq 1, 2$$

and

$$\sum_{n=0}^6 C_n(\psi) r^n = 0 \quad (4.25)$$

$$n \neq 1$$

where

$$B_0(\psi) = \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{4\gamma''(\psi)}{\gamma'^2(\psi)}$$

$$B_3(\psi) = 9m_1 - 6m_1 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{9m_1\rho}{\mu} \frac{1}{\gamma'(\psi)}$$

$$B_4(\psi) = 8m_2^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 32m_2^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{8m_2^2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)}$$

$$B_5(\psi) = 24m_1m_2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 216m_1m_2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1m_2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)}$$

$$B_6(\psi) = 18m_1^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 72m_2^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 279m_1^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36m_1^2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)}$$

$$B_7(\psi) = -360m_1m_2^2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)'$$

$$B_8(\psi) = 16m_2^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 648m_1^2m_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)'$$

$$B_9(\psi) = 96m_1m_2^3 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 378m_1^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)'$$

$$B_{10}(\psi) = 216m_1^2m_2^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$B_{11}(\psi) = 216m_1^3m_2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$B_{12}(\psi) = 81m_1^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$C_0(\psi) = \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)}$$

$$C_2(\psi) = 4m_2 \frac{\beta(\psi)}{\gamma'(\psi)} - \Omega_0$$

$$C_3(\psi) = 9m_1 \frac{\beta(\psi)}{\gamma'(\psi)}$$

$$C_4(\psi) = 4m_2^2 \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - 4m_2^2 \frac{\beta'(\psi)}{\gamma'^2(\psi)}$$

$$C_5(\psi) = 12m_1m_2 \left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \right]$$

$$C_6(\psi) = 9m_1^2 \left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \right]$$

Taking $C_3(\psi) = 0$, we obtain

$$\beta(\psi) = 0$$

Thus, we conclude that this streamline pattern is not permissible for a finitely conducting MHD aligned fluid flow.

Non-MHD Flow

Employing (4.11) and (4.16) in (4.10) with $\Omega = 0$, we have

$$\sum_{n=0}^{12} D_n(\psi) r^n = 0$$

$$n \neq 1, 2$$

This equation is the same as equation (4.24) above with $D_n(\psi) = B_n(\psi)$. Using the consequence of $D_{12}(\psi) = 0$ and $D_9(\psi) = 0$ in $D_3(\psi) = 0$, we get

$$9m_1 - \frac{9m_1\rho}{\mu} \frac{1}{\gamma'(\psi)} = 0$$

which implies that $\gamma'(\psi) = \frac{\rho}{\mu}$. Thus, the unknown function $\gamma(\psi)$ is given by

$$\gamma(\psi) = \frac{\rho}{\mu}\psi + \psi_0$$

where ψ_0 is an arbitrary constant of integration. The exact solutions are given by equations (4.23) with $\beta_0 = 0$. Summing up the above results, we have:

Theorem 4.4. *Streamline pattern $\theta - m_1 r^3 - m_2 r^2 = \text{constant}$ is not permissible for a finitely conducting MHD aligned flow but is permissible for an infinitely conducting MHD aligned and non-MHD flows with solutions given by equations (4.23) and (4.23) with $\beta_0 = 0$ respectively.*

4.3.2. Example II. (Flow with $\theta - ar = \text{constant}$ as streamlines).

We let the family of curves $\theta - ar = \text{constant}$ be the streamlines so that we have

$$\theta = ar + \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (4.26)$$

where $\gamma(\psi)$ is some unknown function of ψ .

Comparing equation (4.26) with (4.8), we get

$$f(r) = ar, \quad g(r) = 1 \quad (4.27)$$

The streamline pattern for this flow is shown in Figure 4.2.

Infinitely Conducting Flow

Using (4.11), (4.12) and (4.27) in equation (4.10), we obtain

$$\sum_{n=0}^4 A_n(\psi) r^n = 0 \quad (4.28)$$

where

$$\begin{aligned}
A_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{2}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
&\quad + \frac{2\mu \beta(\psi) \beta'(\psi)}{\mu^* \gamma'^2(\psi)} \\
A_1(\psi) &= a + 2a \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{a}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
A_2(\psi) &= -a^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
A_3(\psi) &= -2a^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
A_4(\psi) &= a^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Equation (4.28) is a fourth degree polynomial in r with coefficients as functions of ψ only. Since r, ψ are independent variables it follows that equation (4.28) can only hold true for all values of r if the coefficients of different powers of r vanish simultaneously and we have:

$$A_0(\psi) = A_1(\psi) = A_2(\psi) = A_3(\psi) = A_4(\psi) = 0 \quad (4.29)$$

The equation $A_3(\psi) = 0$ holds true in one of the following three cases:

$$\begin{aligned}
\text{i) } a &\neq 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \\
\text{ii) } a &= 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0 \\
\text{iii) } a &= 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0
\end{aligned}$$

We study these three cases separately in the following:

$$\text{Case (i): } \left\{ a \neq 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \right\}.$$

In this case, all the coefficients $A_n(\psi)$, $n = 0, 1, \dots, 4$ vanish simultaneously if

$$\gamma(\psi) = -\frac{1}{\mu} (\rho - \mu^* \beta_0^2) \psi + \psi_0, \quad \beta(\psi) = \beta_0 \quad (4.30)$$

where ψ_0 and $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ are arbitrary constants.

Using equation (4.30) in (4.26), we obtain

$$\theta = ar - \frac{1}{\mu} (\rho - \mu^* \beta_0^2) \psi + \psi_0 \quad (4.31)$$

and the solutions for this flow are given by

$$\begin{aligned} v_1 &= -\frac{\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{r}, & v_2 &= -\frac{a\mu}{[\rho - \mu^* \beta_0^2]} \\ H_1 &= \frac{\mu\beta_0}{[\rho - \mu^* \beta_0^2]} \left[a \sin \theta - \frac{\cos \theta}{r} \right] \\ H_2 &= -\frac{\mu\beta_0}{[\rho - \mu^* \beta_0^2]} \left[a \cos \theta + \frac{\sin \theta}{r} \right] \\ p &= \frac{a^2 \mu^2}{(\rho - \mu^* \beta_0^2)} \ln r - \frac{\rho \mu^2}{2(\rho - \mu^* \beta_0^2)^2 r^2} + p_0 \\ \omega &= -\frac{a\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{r} \\ \Omega &= \beta_0 \omega \end{aligned} \quad (4.32)$$

where p_0 is an arbitrary constant. Thus, we have:

' $\theta - ar = \text{constant}$ ' is a permissible streamline pattern for an infinitely conducting MHD aligned fluid flow and the exact integral for this flow is given by equations (4.32)'.

Case (ii): $\left\{ a = 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0 \right\}.$

In this case, $A_n(\psi) = 0$, $n = 0, 1, \dots, 4$, are identically satisfied if

$$\beta^2(\psi) = \frac{1}{\mu^*} \left[\rho - \mu \gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4\mu \gamma'(\psi) + B\mu \gamma'^2(\psi) \right] \quad (4.33)$$

where B is an arbitrary constant of integration. Equation (4.33) is one equation in two unknowns $\beta(\psi)$ and $\gamma(\psi)$ when $\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0$. There are two ways of getting solutions for this case. One way is to prescribe $\beta(\psi)$ and solve equation (4.33) to get $\gamma(\psi)$ and the second way is to choose a $\gamma(\psi)$ such that $\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0$ and use (4.33) to find $\beta(\psi)$.

The exact solutions, for this flow, are given by

$$\begin{aligned}
v_1 &= \frac{1}{r \gamma'(\psi)}, & v_2 &= 0 \\
H_1 &= \frac{\cos \theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)}, & H_2 &= \frac{\sin \theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)} \\
p &= \frac{1}{2r^2} \left[\frac{\mu}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{\rho}{\gamma'^2(\psi)} \right] + p_0 \\
\omega &= \frac{1}{r^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
\Omega &= \beta(\psi) \omega - \frac{1}{r^2} \frac{\beta'(\psi)}{\gamma'^2(\psi)}
\end{aligned} \tag{4.34}$$

where p_0 is an arbitrary constant and $\beta(\psi)$, $\gamma(\psi)$ are arbitrary functions of ψ such that equation (4.33) is satisfied.

Thus, we have:

' $\theta = \text{constant}$ is a permissible streamline pattern for steady plane rotational infinitely conducting MHD aligned fluid flow and the exact integral for this flow is given by equations (4.34) where $\beta(\psi)$ and $\gamma(\psi)$ are arbitrary functions of ψ such that equation (4.33) and $\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0$ are satisfied'.

As an example, we take $\gamma(\psi) = \frac{6\mu}{\rho\psi}$. With this choice, equation (4.33) gives

$$\beta^2(\psi) = -\frac{1}{\mu^*} \left[\frac{24\mu^2}{\rho\psi^2} + \frac{36\mu^3 B}{\rho^2\psi^4} \right] \tag{4.35}$$

and the solutions (4.34) take the form

$$\begin{aligned}
v_1 &= -\frac{6\mu}{\rho r \theta^2}, & v_2 &= 0 \\
H_1 &= -\frac{6\mu}{\rho r} \beta(\psi) \frac{\cos \theta}{\theta^2}, & H_2 &= -\frac{6\mu}{\rho r} \beta(\psi) \frac{\sin \theta}{\theta^2} \\
p &= p_0 \\
\omega &= -\frac{12\mu}{\rho r^2 \theta^3} \\
\Omega &= \beta(\psi) \omega - \frac{9\mu^2 \beta'(\psi)}{\rho^2 r^2 \theta^4}
\end{aligned}$$

where $\beta(\psi)$ is given by equation (4.35).

$$\text{Case (iii): } \left\{ a = 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \right\}.$$

Integrating $\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0$ three times with respect to ψ , we find that the function $\gamma(\psi)$ is given implicitly by

$$c_1 \gamma^2(\psi) + c_2 \gamma(\psi) + c_3 = \psi \quad (4.36)$$

where c_1 , c_2 and c_3 are arbitrary constants such that c_1 and c_2 are not zero simultaneously. Using (4.36) and $a = 0$ in $A_n(\psi) = 0$, $n = 0, 1, \dots, 4$, we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*} [\rho + 4\mu\gamma'(\psi) - \mu B\gamma'^2(\psi)] \quad (4.37)$$

where $\gamma(\psi)$ is given by equation (4.36) and B is an arbitrary constant of integration.

The exact integral for this flow is given by

$$\begin{aligned} v_1 &= \frac{1}{r} [2c_1 \theta + c_2], & v_2 &= 0 \\ H_1 &= \frac{\beta(\psi)}{r} (2c_1 \theta + c_2) \cos \theta, & H_2 &= \frac{\beta(\psi)}{r} (2c_1 \theta + c_2) \sin \theta \\ p &= p_0 - \frac{\rho}{2r^2} [2c_1 \theta + c_2]^2 \\ \omega &= -\frac{2c_1}{r^2} \\ \Omega &= \beta(\psi)\omega - \frac{\beta'(\psi)}{r^2} (2c_1 \theta + c_2)^2 \end{aligned}$$

where p_0 is an arbitrary constant, $\gamma(\psi)$ and $\beta(\psi)$ are given by (4.36) and (4.37) respectively. Since the pressure must be single-valued, we must take $c_1 = 0$. If $c_1 = 0$, then the flow turns out to be irrotational.

Finitely Conducting Flow

Using (4.11) and (4.27) in (4.13) and (4.14), we get

$$\sum_{n=0}^4 B_n(\psi) r^n = 0 \quad (4.38)$$

and

$$\sum_{n=0}^2 C_n(\psi) r^n = 0 \quad (4.39)$$

where

$$\begin{aligned} B_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ B_1(\psi) &= a + 2a \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{a\rho}{\mu} \frac{1}{\gamma'(\psi)} \\ B_2(\psi) &= -a^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ B_3(\psi) &= -2a^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\ B_4(\psi) &= a^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ C_0(\psi) &= \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \\ C_1(\psi) &= \frac{a\beta(\psi)}{\gamma'(\psi)} \\ C_2(\psi) &= a^2 \frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{a^2\beta'(\psi)}{\gamma'^2(\psi)} - \Omega_0 \end{aligned}$$

Equations (4.38) and (4.39) must hold true for all values of r . Since r, ψ are independent variables, then we have

$$B_0(\psi) = B_1(\psi) = B_2(\psi) = B_3(\psi) = B_4(\psi) = C_0(\psi) = C_1(\psi) = C_2(\psi) = 0$$

Requiring $C_1(\psi) = 0$, we have

$$a = 0$$

since $\beta(\psi) \neq 0$. Using $a = 0$, the equations $B_1(\psi) = 0$, $B_2(\psi) = 0$, $B_3(\psi) = 0$, $B_4(\psi) = 0$ are identically satisfied and $B_0(\psi) = 0$, $C_0(\psi) = 0$, $C_2(\psi) = 0$ give

$$\begin{aligned} \frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{4}{\gamma'(\psi)} - \frac{\rho}{\mu} \frac{1}{\gamma'^2(\psi)} &= \psi_0 \\ \beta^2(\psi) &= \beta_0 \gamma'^2(\psi) \end{aligned} \quad (4.40)$$

$$\Omega_0 = 0$$

where ψ_0 and β_0 are arbitrary constants of integration. Proceeding as in infinitely conducting flow, we find that the exact solutions for this flow are given by

$$\begin{aligned}
v_1 &= \frac{1}{r \gamma'(\psi)}, & v_2 &= 0 \\
H_1 &= \frac{\cos \theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)}, & H_2 &= \frac{\sin \theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)} \\
p &= \frac{1}{2r^2} \left[\frac{\mu}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{\rho}{\gamma'^2(\psi)} \right] + p_0 \\
\omega &= \frac{1}{r^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)}
\end{aligned} \tag{4.41}$$

where p_0 is an arbitrary constant and $\gamma(\psi)$, $\beta(\psi)$ are given by equations (4.40). Thus, we have:

' $\theta = \text{constant}$ is a permissible streamline pattern for a finitely conducting MHD aligned fluid flow and the exact solutions of this flow are given by equations (4.41) with $\gamma(\psi)$ and $\beta(\psi)$ given by equations (4.40)'.

As an example, we let $\gamma(\psi) = \psi$ and equations (4.40) and (4.41) give

$$\begin{aligned}
\psi_0 &= -4 - \frac{\rho}{\mu}, & \beta^2(\psi) &= \beta_0 \\
v_1 &= \frac{1}{r}, & v_2 &= 0 \\
H_1 &= \frac{\sqrt{\beta_0} \cos \theta}{r}, & H_2 &= \frac{\sqrt{\beta_0} \sin \theta}{r} \\
p &= p_0 - \frac{\rho}{2r^2}, & \omega &= 0
\end{aligned}$$

where β_0 and p_0 are arbitrary constants.

Non-MHD Flow

Employing (4.11) and (4.27) in (4.13), we obtain

$$\sum_{n=0}^4 D_n(\psi) r^n = 0 \tag{4.42}$$

where

$$\begin{aligned}
D_0(\psi) &= \frac{4\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
D_1(\psi) &= a + 2a \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{a\rho}{\mu} \frac{1}{\gamma'(\psi)} \\
D_2(\psi) &= -a^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2a^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
D_3(\psi) &= -2a^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
D_4(\psi) &= a^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Since r and ψ are independent variables and equation (4.42) is a fourth degree polynomial with coefficients as functions of ψ only, then we must have

$$D_0(\psi) = D_1(\psi) = D_2(\psi) = D_3(\psi) = D_4(\psi) = 0 \quad (4.43)$$

Requiring $D_3(\psi) = 0$, we get the following three cases:

$$\begin{aligned}
\text{i) } a &\neq 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \\
\text{ii) } a &= 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0 \\
\text{iii) } a &= 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0
\end{aligned}$$

We study these three cases separately as follows:

$$\text{Case (i): } \left\{ a \neq 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \right\}.$$

Using $\left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0$, in $D_0(\psi) = 0$, $D_1(\psi) = 0$, $D_2(\psi) = 0$ and in $D_4(\psi) = 0$, we have

$$\gamma(\psi) = -\frac{\rho}{\mu}\psi + \psi_0$$

where ψ_0 is an arbitrary constant. In this case solutions are given by equations (4.32) with $\beta_0 = 0$.

$$\text{Case (ii): } \left\{ a = 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \neq 0 \right\}.$$

With $a = 0$, equations $D_1(\psi) = 0$, $D_2(\psi) = 0$ and $D_4(\psi) = 0$ are identically satisfied and $D_0(\psi) = 0$ gives

$$\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{4}{\gamma'(\psi)} - \frac{\rho}{\mu} \frac{1}{\gamma'^2(\psi)} = \psi_0$$

where ψ_0 is an arbitrary constant. For this case, exact solutions are given by equations (4.34) with $\beta(\psi) = 0$.

$$\text{Case (iii): } \left\{ a = 0, \quad \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' = 0 \right\}.$$

For this case, equations (4.43) are identically satisfied if

$$\gamma(\psi) = c_1\psi + c_2$$

where $c_1 \neq 0$ and c_2 are arbitrary constants. The solutions for this case are given by

$$\begin{aligned} v_1 &= \frac{1}{c_1 r}, \quad v_2 = 0 \\ p &= p_0 - \frac{\rho}{2c_1^2 r^2} \\ \omega &= 0 \end{aligned} \tag{4.44}$$

where p_0 is an arbitrary constant.

Summing up, we have:

Theorem 4.5. *Streamline pattern $\theta - ar = \text{constant}$ in a steady plane motion is permissible for an infinitely conducting MHD aligned and non-MHD fluid flow. It is also permissible for a finitely conducting MHD aligned flow if $a = 0$.*

4.3.3 Example III. (Flow with $\theta - a_1 r^m - a_2 \ln r = \text{constant}$ as streamlines).

We assume

$$\theta = a_1 r^m + a_2 \ln r + \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{4.45}$$

where a_1 , a_2 and m are arbitrary constants and, therefore, we have

$$f(r) = a_1 r^m + a_2 \ln r, \quad g(r) = 1 \quad (4.46)$$

The streamline pattern for this flow is shown in Figure 4.3.

Infinitely Conducting Flow

Proceeding as in the previous example, we find that the functions $\gamma(\psi)$ and $\beta(\psi)$ must satisfy

$$\sum_{n=0}^4 A_n(\psi) r^{nm-3} = 0 \quad (4.47)$$

where

$$\begin{aligned} A_0(\psi) &= (1 + a_2^2)^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 4a_2 (1 + a_2^2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\ &\quad + 4(1 + a_2^2) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2}{\mu} (1 + a_2^2) \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ A_1(\psi) &= 4ma_1a_2 (1 + a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + a_1a_2m (2m^2 - 7m + 10) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ &\quad - 2a_1m(m-2)(1 + 3a_2^2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + a_1m^2(m-2)^2 \\ &\quad - \frac{1}{\mu} a_1m(m-2) [\rho - \mu^* \beta^2(\psi)] \left[\frac{m}{\gamma'(\psi)} + a_2 \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right] \\ &\quad - \frac{\mu^*}{\mu} a_1a_2m(m-2) \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\ A_2(\psi) &= 2a_1^2m^2 (1 + 3a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 12a_1^2a_2m^2(m-1) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\ &\quad + a_1^2m^2 (5m^2 - 6m + 4) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ &\quad - \frac{2}{\mu} a_1^2m^2(m-1) \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ A_3(\psi) &= 4a_1^3a_2m^3 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 2a_1^3m^3(3m-2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\ A_4(\psi) &= a_1^2m^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \end{aligned}$$

The polynomial (4.47) in r , with coefficients as functions of ψ only, holds true provided all coefficients vanish simultaneously and we have

$$A_0(\psi) = A_1(\psi) = A_2(\psi) = A_3(\psi) = A_4(\psi) = 0 \quad (4.48)$$

In particular, $A_1(\psi) = 0$, $A_2(\psi) = 0$, $A_3(\psi) = 0$ and $A_4(\psi) = 0$ are identically satisfied in one of the following four cases:

- a) $a_1 \neq 0$, $m \neq 2$, $\gamma'(\psi) = \frac{1}{\mu(m-2)} (\rho - \mu^* \beta_0^2)$, $\beta(\psi) = \beta_0$, $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$
- b) $\gamma''(\psi) = 0$, $a_1 \neq 0$, $m = 2$, $\beta(\psi) = \beta_0$
- c) $a_1 = 0$, $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0$
- d) $a_1 = 0$, $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0$

We study these four cases separately as follows:

Case (a): $\left\{ a_1 \neq 0, m \neq 2, \gamma'(\psi) = \frac{1}{\mu(m-2)} (\rho - \mu^* \beta_0^2), \beta(\psi) = \beta_0, \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}} \right\}.$

In this case, $A_0(\psi) = 0$ is identically satisfied and we have

$\theta - a_1 r^m - a_2 \ln r = \text{constant}$ with $m \neq 2$ is a permissible streamline pattern and

the exact integral for this flow are given by

$$\begin{aligned}
v_1 &= \frac{\mu(m-2)}{r[\rho - \mu^* \beta_0^2]}, & v_2 &= \frac{\mu(m-2)}{\rho - \mu^* \beta_0^2} \left[m a_1 r^{m-1} + \frac{a_2}{r} \right] \\
H_1 &= \frac{\mu \beta_0 (m-2)}{\rho - \mu \beta_0^2} \left[\frac{\cos \theta - a_2 \sin \theta}{r} - m a_1 r^{m-1} \sin \theta \right] \\
H_2 &= \frac{\mu \beta_0 (m-2)}{\rho - \mu \beta_0^2} \left[\frac{\sin \theta + a_2 \cos \theta}{r} + m a_1 r^{m-1} \cos \theta \right] \\
p &= \begin{cases} \frac{\mu^2 a_1 m^2 (m-2)^2}{\rho - \mu^* \beta_0^2} \left[\frac{a_1 m}{2(m-1)} r^{2m-2} + \frac{a_2}{m-2} r^{m-2} \right] \\ - \frac{1}{2} \rho \frac{\mu^2 (m-2)^2}{(\rho - \mu^* \beta_0^2)^2} \left\{ \frac{1+a_2^2}{r^2} + m^2 a_1^2 r^{2m-2} + 2m a_1 a_2 r^{m-2} \right\} \\ + p_0; & m \neq 1 \\ \\ \frac{\mu^2 a_1}{\rho - \mu^* \beta_0^2} \left[a_1 \ln r - \frac{a_2}{r} \right] - \frac{1}{2} \rho \frac{\mu^2}{(\rho - \mu^* \beta_0^2)^2} \left\{ \frac{1+a_2^2}{r^2} + \frac{2a_1 a_2}{r} \right\} \\ + p_0; & m = 1 \end{cases} \quad (4.49) \\
\omega &= \frac{\mu m^2 (m-2) a_1}{\rho - \mu \beta_0^2} r^{m-2} \\
\Omega &= \beta_0 \omega
\end{aligned}$$

where p_0 is an arbitrary constant of integration and $m \neq 2$.

Case (b): $\{\gamma''(\psi) = 0, \quad a_1 \neq 0, \quad m = 2, \quad \beta(\psi) = \beta_0\}$.

Since $\gamma''(\psi) = 0$, then we get

$$\gamma(\psi) = b_1 \psi + b_2 \quad (4.50)$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. Using $\beta(\psi) = \beta_0$ and (4.50), $A_0(\psi)$ is identically satisfied and we have

$\theta - a_1 r^2 - a_2 \ln r = \text{constant}$ is a possible streamline pattern and the exact integral

associated with this flow is given by

$$\begin{aligned}
v_1 &= \frac{1}{b_1 r}, & v_2 &= \frac{1}{b_1} \left[2a_1 r + \frac{a_2}{r} \right] \\
H_1 &= \frac{\beta_0}{b_1} \left[\frac{\cos\theta - a_2 \sin\theta}{r} - 2a_1 r \sin\theta \right] \\
H_2 &= \frac{\beta_0}{b_1} \left[\frac{\sin\theta + a_2 \cos\theta}{r} + 2a_1 r \cos\theta \right] \\
p &= p_0 - \frac{4a_1}{b_1^2} (\rho - \mu^* \beta_0^2) [\theta - a_1 r^2 - a_2 \ln r - b_2] \\
&\quad - \frac{1}{2} \frac{\rho}{b_1^2} \left[\frac{1 + a_2^2}{r^2} + 4a_1^2 r^2 + 4a_1 a_2 \right] \\
\omega &= \frac{4a_1}{b_1} \\
\Omega &= \beta_0 \omega
\end{aligned} \tag{4.51}$$

where p_0 is an arbitrary constant'. Since the pressure function p must be single-valued, we must take $\beta_0^2 = \frac{\rho}{\mu^*}$.

$$\text{Case (c): } \left\{ a_1 = 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0 \right\}.$$

All coefficients $A_n(\psi)$, $n = 0, 1, \dots, 4$ vanish simultaneously if

$$\beta^2(\psi) = \frac{\mu}{\mu^*} \left\{ 4\gamma'(\psi) - (1 + a_2^2) \gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - 4a_2 \frac{\gamma''(\psi)}{\gamma'(\psi)} - b_3 \gamma'^2(\psi) + \frac{\rho}{\mu} \right\} \tag{4.52}$$

where b_3 is an arbitrary constant and $\gamma(\psi)$ is an arbitrary function of ψ such that $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0$. Thus,

$\theta - a_2 \ln r = \text{constant}$ can serve as streamline pattern and the exact integral for

this rotational flow is given by

$$\begin{aligned}
v_1 &= \frac{1}{r \gamma'(\psi)}, & v_2 &= \frac{a_2}{r \gamma'(\psi)} \\
H_1 &= \frac{\cos\theta - a_2 \sin\theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)}, & H_2 &= \frac{\sin\theta + a_2 \cos\theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)} \\
p &= \frac{\mu}{2} (1 + a_2^2) \frac{1}{r^2} \left[2a_2 \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + (1 + a_2^2) \frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right] \\
&\quad - \frac{1}{2} \frac{\rho}{\gamma'^2(\psi)} \frac{1 + a_2^2}{r^2} + p_0 \\
\omega &= \frac{1 + a_2^2}{r^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
\Omega &= \beta(\psi)\omega - \frac{1 + a_2^2}{r^2} \frac{\beta'(\psi)}{\gamma'^2(\psi)}
\end{aligned} \tag{4.53}$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by (4.52) and $\gamma(\psi)$ is an arbitrary function of ψ .

$$\text{Case (d): } \left\{ a_1 = 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \right\}.$$

Integrating $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0$ four times with respect to ψ , we obtain

$$c_1 \gamma^3(\psi) + c_2 \gamma^2(\psi) + c_3 \gamma(\psi) + c_4 = \psi \tag{4.54}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration such that c_1, c_2 and c_3 are not zero simultaneously. Using $a_1 = 0$ and $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0$ in $A_0(\psi) = 0$ and integrating the resulting equation with respect to ψ , we get

$$\beta^2(\psi) = \frac{\mu}{\mu^*} \left[4\gamma'(\psi) - 4a_2 \frac{\gamma''(\psi)}{\gamma'(\psi)} + \frac{\rho}{\mu} + c_5 \gamma'^2(\psi) \right] \tag{4.55}$$

where c_5 is an arbitrary constant of integration and $\gamma(\psi)$ is given implicitly by (4.54).

The exact solutions for this flow are given by

$$\begin{aligned}
v_1 &= \frac{1}{r} \left[3c_1 (\theta - a_2 \ln r)^2 + 2c_2 (\theta - a_2 \ln r) + c_3 \right] \\
v_2 &= \frac{a_2}{r} \left[3c_1 (\theta - a_2 \ln r)^2 + 2c_2 (\theta - a_2 \ln r) + c_3 \right] \\
H_1 &= \frac{\cos \theta - a_2 \sin \theta}{r} \beta(\psi) \left[3c_1 (\theta - a_2 \ln r)^2 + 2c_2 (\theta - a_2 \ln r) + c_3 \right] \\
H_2 &= \frac{\sin \theta + a_2 \cos \theta}{r} \beta(\psi) \left[3c_1 (\theta - a_2 \ln r)^2 + 2c_2 (\theta - a_2 \ln r) + c_3 \right] \\
p &= (1 + a_2^2) \frac{1}{r^2} \left\{ -\mu a_2 [6c_1 (\theta - a_2 \ln r) + 2c_2] - 3\mu (1 + a_2^2) c_1 \right. \\
&\quad \left. - \rho \left[9c_1^2 (\theta - a_2 \ln r)^4 + 12c_1 c_2 (\theta - a_2 \ln r)^3 + 4c_2 c_3 (\theta - a_2 \ln r) \right. \right. \\
&\quad \left. \left. + (6c_1 c_3 + 4c_2^2) (\theta - a_2 \ln r)^2 + c_3^2 \right] \right\} + p_0 \\
\omega &= -\frac{1 + a_2^2}{r^2} [6c_1 (\theta - a_2 \ln r) + 2c_2] \\
\Omega &= \beta(\psi) \omega - \frac{1 + a_2^2}{r^2} \beta'(\psi) \left[3c_1 (\theta - a_2 \ln r)^2 + 2c_2 (\theta - a_2 \ln r) + c_3 \right]^2
\end{aligned} \tag{4.56}$$

where p_0 is an arbitrary constant of integration and $\beta(\psi)$ is given by (4.55). Since the pressure function must be single-valued, we must take $c_1 = c_2 = 0$. If $c_1 = c_2 = 0$, then $\omega = 0$ and the flow is irrotational.

Finitely Conducting Flow

Employing (4.11) and (4.46) in (4.13) and (4.14), we get

$$\sum_{n=0}^4 B_n(\psi) r^{nm-3} = 0 \tag{4.57}$$

and

$$\sum_{n=0}^2 C_n(\psi) r^{nm-2} + C_3(\psi) = 0 \tag{4.58}$$

where

$$\begin{aligned}
B_0(\psi) &= (1 + a_2^2)^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 4a_2 (1 + a_2^2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + 4(1 + a_2^2) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
B_1(\psi) &= 4ma_1a_2 (1 + a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + a_1a_2m (2m^2 - 7m + 10) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad - 2a_1m(m-2) (1 + 3a_2^2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + a_1m^2(m-2)^2 \\
&\quad - \frac{a_1m\rho}{\mu} (m-2) \left[\frac{m}{\gamma'(\psi)} + a_2 \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right] \\
B_2(\psi) &= 2a_1^2m^2 (1 + 3a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 12a_1^2a_2m^2 (m-1) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + a_1^2m^2 (5m^2 - 6m + 4) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2\rho}{\mu} a_1^2m^2 (m-1) \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
B_3(\psi) &= 4a_1^3a_2m^3 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 2a_1^3m^3 (3m-2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
B_4(\psi) &= a_1^2m^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
C_0(\psi) &= (1 + a_2^2) \left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \right] \\
C_1(\psi) &= a_1m^2 \frac{\beta(\psi)}{\gamma'(\psi)} + 2a_1a_2m \left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \right] \\
C_2(\psi) &= a_1^2m^2 \left[\frac{\beta(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\beta'(\psi)}{\gamma'^2(\psi)} \right] \\
C_3(\psi) &= -\Omega_0
\end{aligned}$$

Requiring (4.57) and (4.58) to hold true for all values of r , we get

$$B_i(\psi) = C_j(\psi) = 0, \quad i = 0, 1, 2, 3, 4 \quad \text{and} \quad j = 0, 1, 2, 3$$

From $C_3(\psi) = 0$ and $C_0(\psi) = 0$, we have

$$\Omega_0 = 0 \quad \text{and} \quad \beta(\psi) = k_1\gamma'(\psi) + k_2 \quad (4.59)$$

respectively where k_1 and k_2 are arbitrary constants of integration. Using (4.59) in $C_1(\psi) = 0$, we obtain

$$a_1 = 0 \quad (4.60)$$

Employing (4.60), we find that $B_i(\psi) = 0$, $i = 0, 1, 2, 3, 4$ are identically satisfied if

$$(1 + a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 4a_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} = 0 \quad (4.61)$$

Thus, the exact solutions for finitely conducting flow are given by

$$\begin{aligned} v_1 &= \frac{1}{r \gamma'(\psi)}, & v_2 &= \frac{a_2}{r \gamma'(\psi)} \\ H_1 &= \frac{\cos\theta - a_2 \sin\theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)}, & H_2 &= \frac{\sin\theta + a_2 \cos\theta}{r} \frac{\beta(\psi)}{\gamma'(\psi)} \\ p &= \frac{\mu}{2r^2} (1 + a_2^2) \left[2a_2 \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + (1 + a_2^2) \frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right] \\ &\quad - \frac{1}{2} \rho (1 + a_2^2) \frac{1}{r^2} \frac{1}{\gamma'^2(\psi)} + p_0 \\ \omega &= \frac{1 + a_2^2}{r^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ \Omega &= \beta(\psi) \omega - \frac{1 + a_2^2}{r^2} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \end{aligned}$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by equation (4.59) and $\gamma(\psi)$ is given by equation (4.61). Since the pressure function should be single-valued, the function $\gamma(\psi)$ must also satisfy the following equation

$$2\mu a_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \mu (1 + a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{2\rho \gamma''(\psi)}{\gamma'^3(\psi)} = 0$$

Using this equation and (4.61), we find that $\gamma''(\psi) = 0$ and the flow is irrotational.

Non-MHD Flow

Substituting (4.11) and (4.46) in (4.13), we obtain

$$\sum_{n=0}^4 D_n(\psi) r^{nm-3} = 0 \quad (4.62)$$

where $D_n(\psi)$ are given by $B_n(\psi)$ of the finitely conducting case. The coefficients $D_n(\psi)$, $n = 1, \dots, 4$ vanish simultaneously in one of the following four cases:

$$\begin{aligned} \text{a) } a_1 &\neq 0, \quad m \neq 2, \quad \gamma'(\psi) = \frac{\rho}{\mu(m-2)} \\ \text{b) } a_1 &\neq 0, \quad m = 2, \quad \gamma''(\psi) = 0 \\ \text{c) } a_1 &= 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0 \\ \text{d) } a_1 &= 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \end{aligned}$$

We study these four cases separately in the following:

Case (a): $\left\{ a_1 \neq 0, \quad m \neq 2, \quad \gamma'(\psi) = \frac{\rho}{\mu(m-2)} \right\}.$

In this case $D_0(\psi) = 0$ is also satisfied and the exact integral is given by equations (4.49) with $\beta_0 = 0$.

Case (b): $\{ a_1 \neq 0, \quad m = 2, \quad \gamma''(\psi) = 0 \}.$

Since $\gamma''(\psi) = 0$, we get

$$\gamma(\psi) = b_1\psi + b_2$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. Using $\gamma''(\psi) = 0$, $D_0(\psi) = 0$ is identically satisfied and the exact solutions are given by equations (4.51) with $\beta_0 = 0$.

Case (c): $\left\{ a_1 = 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \right\}.$

All coefficients $D_n(\psi)$, $n = 0, 1, \dots, 4$ vanish simultaneously if

$$(1 + a_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 4a_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2\rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} = 0 \quad (4.63)$$

and the exact solutions are given by equations (4.53) with $\beta(\psi) = 0$ and $\gamma(\psi)$ given by equation (4.63).

Case (d): $\left\{ a_1 = 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \right\}.$

Integrating $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0$ four times with respect to ψ , we obtain

$$c_1 \gamma^3(\psi) + c_2 \gamma^2(\psi) + c_3 \gamma(\psi) + c_4 = \psi \quad (4.64)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants such that c_1, c_2 and c_3 are not equal to zero simultaneously. Using (4.64) in $D_0(\psi) = 0$, we get

$$c_1 = c_2 = 0$$

and the exact integral for this irrotational flow is given by equations (4.56) with $\beta(\psi) = 0$ and $\gamma'(\psi) = \frac{1}{c_3}$.

Summing up, we have

Theorem 4.6. *The streamline pattern $\theta - a_1 r^m - a_2 \ln r = \text{constant}$ is permissible for an infinitely conducting MHD aligned, a finitely conducting MHD aligned with $a_1 = 0$ and for non-MHD fluid flows.*

4.3.4 Example IV. (Flow with $r^2(\theta - \ln r) = \text{constant}$ as streamlines). We assume that

$$\theta = \ln r + \frac{1}{r^2} \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (4.65)$$

where $\gamma(\psi)$ is an arbitrary function of ψ . Comparing (4.65) with (4.8), we get

$$f(r) = \ln r, \quad g(r) = \frac{1}{r^2}$$

This streamline pattern is shown in Figure 4.4.

Infinitely Conducting Flow

Proceeding as in the examples above, we find that the functions $\gamma(\psi)$ and $\beta(\psi)$ must satisfy

$$\sum_{n=0}^4 A_{2n}(\psi) r^{2n} = 0 \quad (4.66)$$

where

$$\begin{aligned}
A_0(\psi) &= 16\gamma^4(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 64\gamma^3(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 32 \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\
A_2(\psi) &= -32\gamma^3(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 128\gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - 80 \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{8}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \left[\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} - \frac{\gamma(\psi)}{\gamma'(\psi)} \right] + \mu^* \frac{\gamma^2(\psi)\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_4(\psi) &= 32\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 112\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 56 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
A_6(\psi) &= -16\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 32 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad - \frac{4}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_8(\psi) &= 4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Since $A_{2n}(\psi)$, $n = 0, 1, \dots, 4$ are functions of ψ only and r, ψ are independent variables, then equation (4.66) holds true for all values of r if

$$A_0(\psi) = A_2(\psi) = A_4(\psi) = A_6(\psi) = A_8(\psi) = 0$$

These five equations are satisfied simultaneously if

$$\gamma(\psi) = b_1\psi + b_2, \quad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}}$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. The exact solutions for flow are given

by

$$\begin{aligned}
v_1 &= \frac{r}{b_1}, & v_2 &= -\frac{1}{b_1} [2r(\theta - \ln r) - r] \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}} \frac{1}{b_1} [r \cos \theta - r \sin \theta (2 \ln r - 2\theta + 1)] \\
H_2 &= \sqrt{\frac{\rho}{\mu^*}} \frac{1}{b_1} [r \sin \theta + r \cos \theta (2 \ln r - 2\theta + 1)] \\
p &= \frac{2\mu}{b_1} [2\theta + 2 \ln r] - \frac{1}{2} \frac{\rho}{b_1^2} r^2 [1 + (2 \ln r - 2\theta + 1)^2] + p_0 \\
\omega &= \frac{4}{b_1} [1 - \theta + \ln r] \\
\Omega &= \sqrt{\frac{\rho}{\mu^*}} \omega
\end{aligned} \tag{4.67}$$

where p_0 is an arbitrary constant.

Finitely Conducting and non-MHD Flows

Proceeding as in the previous examples, we find that for both finitely conducting and non-MHD flows $\gamma(\psi) = 0$. Thus, this family of curves is not a permissible pattern for finitely conducting MHD aligned and non-MHD flows.

4.3.5 Example V. (Flow with $\theta - e^{mr} = \text{constant}$ as streamline pattern).

In this example, we show that infinitely conducting fluid cannot flow along the family of curves $\theta - e^{mr} = \text{constant}$ where $m \neq 0$ is an arbitrary constant. To prove this claim, we assume the contrary to arrive at a contradiction. We assume that

$$\theta - e^{mr} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{4.68}$$

where $\gamma(\psi)$ is an arbitrary function of ψ . Comparing (4.68) with (4.8), we find that

$$f(r) = e^{mr}, \quad g(r) = 1 \tag{4.69}$$

Using (4.11), (4.12) and (4.69) in equation (4.10), we obtain

$$\begin{aligned}
B_0(\psi) + \sum_{n=1}^4 B_n(\psi) r^n e^{mr} + \sum_{n=2}^4 B_{3+n}(\psi) r^n e^{2mr} + \sum_{n=3}^4 B_{5+n}(\psi) r^n e^{3mr} \\
+ B_{10}(\psi) r^4 e^{4mr} = 0
\end{aligned} \tag{4.70}$$

where

$$\begin{aligned}
B_0(\psi) &= \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{2}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
&\quad + \frac{2\mu^*}{\mu} \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\
B_1(\psi) &= 2m \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + m + \frac{m}{\mu\gamma'(\psi)} [\rho - \mu^* \beta^2(\psi)] \\
B_2(\psi) &= -2m^2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - m^2 - \frac{m^2}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
B_3(\psi) &= 2m^3 - \frac{m^3}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
B_4(\psi) &= m^4 \\
B_5(\psi) &= 2m^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - m^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
B_6(\psi) &= 6m^3 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2m^3}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
B_7(\psi) &= m^4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
B_8(\psi) &= -2m^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
B_9(\psi) &= -4m^4 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
B_{10}(\psi) &= m^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Since r, ψ are independent variables and $B_i(\psi)$, $i = 0, 1, \dots, 10$ are functions of ψ only, it follows from (4.70) that the coefficients $B_i(\psi)$, $i = 0, 1, \dots, 10$ must vanish simultaneously. Taking $B_4(\psi) = 0$, we get $m = 0$ which contradicts our assumption that $m \neq 0$. Therefore, we conclude that infinitely conducting fluid cannot flow along the family of curves $\theta - e^{mr} = \text{constant}$.

Proceeding as in the last example, we can show that none of the three types of fluid motion considered is possible for many families of curves that belong to the form $\frac{\theta - f(r)}{g(r)} = \text{constant}$. Some examples are $\theta - r e^{mr} = \text{constant}$, $\theta - e^{r^2} =$

constant and $r^3\theta = \text{constant}$.

More exact solutions for infinitely conducting MHD aligned flow are provided in the following examples.

4.3.6. Example VI. (Flow with $\theta - r^m = \text{constant}$ as streamlines).

We let

$$\theta - r^m = \gamma(\psi)$$

and we find that

$$\begin{aligned} \gamma(\psi) &= \begin{cases} \frac{1}{\mu(m-2)} [\rho - \mu^* \beta_0^2] \psi + \psi_0, & m \neq 2 \\ a\psi + b, & m = 2 \end{cases} \\ \beta(\psi) &= \begin{cases} \beta_0, & \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}, & m \neq 2 \\ \beta_0, & m = 2 \end{cases} \\ v_1 &= \begin{cases} \frac{\mu(m-2)}{[\rho - \mu^* \beta_0^2] r}, & m \neq 2 \\ \frac{1}{ar}, & m = 2 \end{cases} \\ v_2 &= \begin{cases} \frac{\mu m(m-2)}{\rho - \mu^* \beta_0^2} r^{m-1}, & m \neq 2 \\ \frac{2r}{a}, & m = 2 \end{cases} \end{aligned}$$

$$\begin{aligned}
H_1 &= \begin{cases} \frac{\mu(m-2)\beta_0}{\rho - \mu^*\beta_0^2} \left[\frac{\cos\theta}{r} - mr^{m-1}\sin\theta \right], & m \neq 2 \\ \frac{1}{a} \left[\frac{\cos\theta}{r} - 2r\sin\theta \right], & m = 2 \end{cases} \\
H_2 &= \begin{cases} \frac{\mu(m-2)\beta_0}{\rho - \mu^*\beta_0^2} \left[\frac{\sin\theta}{r} + mr^{m-1}\cos\theta \right], & m \neq 2 \\ \frac{1}{a} \left[\frac{\sin\theta}{r} + 2r\cos\theta \right], & m = 2 \end{cases} \\
p &= \begin{cases} \frac{\mu^2 m^3 (m-2)^2}{2(m-1)(\rho - \mu^*\beta_0^2)} r^{2m-2} - \frac{\rho\mu^2(m-2)^2}{2(\rho - \mu^*\beta_0^2)^2} \left[\frac{1}{r^2} + m^2 r^{2m-2} \right] \\ \quad + p_0, & m \neq 2, \quad m \neq 1 \\ \frac{\mu^2}{\rho - \mu^*\beta_0^2} \ln r - \frac{\rho\mu^2}{2(\rho - \mu^*\beta_0^2)^2 r^2} + p_0, & m \neq 2, \quad m = 1 \\ p_0 - \frac{4}{a^2} (\rho - \mu^*\beta_0^2) [\theta - r^2 - b] - \frac{\rho}{2a^2} \left[\frac{1}{r^2} + 4r^2 \right], & m = 2 \end{cases} \\
\omega &= \begin{cases} \frac{\mu m(m-2)}{\rho - \mu^*\beta_0^2} r^{m-2}, & m \neq 2 \\ \frac{4}{a}, & m = 2 \end{cases} \\
\Omega &= \beta_0 \omega
\end{aligned}$$

where $\psi_0, \beta_0 \neq 0, a \neq 0, b$ and p_0 are arbitrary constants. The streamline pattern is given in Figure 4.5.

4.3.7. Example VII. (Flow with $\theta - a_1 r^2 - a_2 r = \text{constant}$ as streamlines).

We take

$$\theta - a_1 r^2 - a_2 r = \gamma(\psi)$$

and we have

$$\begin{aligned}
\gamma(\psi) &= -\frac{1}{\mu} [\rho - \mu^* \beta_0^2] \psi + \psi_0 \\
\beta(\psi) &= \beta_0, \quad \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}} \\
v_1 &= -\frac{\mu}{[\rho - \mu^* \beta_0^2] r} \\
v_2 &= -\frac{\mu}{[\rho - \mu^* \beta_0^2]} (2a_1 r + a_2) \\
H_1 &= -\frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]} \left[\frac{\cos \theta}{r} - (2a_1 r + a_2) \sin \theta \right] \\
H_2 &= -\frac{\mu \beta_0}{[\rho - \mu^* \beta_0^2]} \left[\frac{\sin \theta}{r} + (2a_1 r + a_2) \cos \theta \right]
\end{aligned}$$

$$\begin{aligned}
p &= \frac{\mu^2}{[\rho - \mu^* \beta_0^2]} [6a_1 a_2 r + a_2^2 \ln r - 4a_1 (\theta - a_1 r^2 - \psi_0)] \\
&\quad - \frac{\rho \mu^2}{2[\rho - \mu^* \beta_0^2]^2} \left[\frac{1}{r^2} + 4a_1^2 r^2 + 4a_1 a_2 r + a_2^2 \right] + p_0 \\
\omega &= -\frac{\mu}{[\rho - \mu^* \beta_0^2]} \left[4a_1 + \frac{a_2}{r} \right] \\
\Omega &= \beta_0 \omega
\end{aligned}$$

where $\psi_0, \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ and p_0 are arbitrary constants. Figure 4.6 shows the flow pattern for this example.

4.3.8. Example VIII. (Flow with $\theta - a_1 (\ln r)^2 - a_2 \ln r = \text{constant}$ as stream-lines).

We assume that

$$\theta - a_1 (\ln r)^2 - a_2 \ln r = \gamma(\psi)$$

and we have

$$\begin{aligned}
\gamma(\psi) &= -\frac{1}{2\mu} [\rho - \mu^* \beta_0^2] \psi + \psi_0 \\
\beta(\psi) &= \beta_0, \quad \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}} \\
v_1 &= -\frac{2\mu}{[\rho - \mu^* \beta_0^2] r} \\
v_2 &= -\frac{2\mu}{[\rho - \mu^* \beta_0^2]} \left[\frac{2a_1 \ln r}{r} + \frac{a_2}{r} \right] \\
H_1 &= -\frac{2\mu\beta_0}{[\rho - \mu^* \beta_0^2] r} [\cos\theta - (2a_1 \ln r + a_2) \sin\theta] \\
H_2 &= -\frac{2\mu\beta_0}{[\rho - \mu^* \beta_0^2] r} [\sin\theta + (2a_1 \ln r + a_2) \cos\theta]
\end{aligned}$$

$$\begin{aligned}
p &= -\frac{4\mu^2 a_1}{[\rho - \mu^* \beta_0^2] r^2} [2a_1 \ln r + a_1 + a_2] - \frac{2\mu^2 \rho}{[\rho - \mu^* \beta_0^2]^2 r^2} [1 + (2a_1 \ln r + a_2)^2] + p_0 \\
\omega &= -\frac{4\mu a_1}{[\rho - \mu^* \beta_0^2] r^2} \\
\Omega &= \beta_0 \omega
\end{aligned}$$

where p_0 and $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ are arbitrary constants. The flow pattern is shown in Figure 4.7.

4.3.9. Example IX. (Flow with $r^2\theta = \text{constant}$ as streamlines).

We let

$$r^2\theta = \gamma(\psi)$$

and we have

$$\begin{aligned}
\gamma(\psi) &= a_1\psi + a_2 \\
\beta(\psi) &= \sqrt{\frac{\rho}{\mu^*}} \\
v_1 &= \frac{r}{a_1}, \quad v_2 = -\frac{2r\theta}{a_1} \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}} \frac{r}{a_1} [\cos\theta + 2\theta \sin\theta], \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} \frac{r}{a_1} [\sin\theta - 2\theta \cos\theta] \\
p &= \frac{4\mu}{a_1} \ln r - \frac{\rho r^2}{2a_1^2} (1 + 4\theta^2) + p_0 \\
\omega &= -\frac{4\theta}{a_1} \\
\Omega &= \sqrt{\frac{\rho}{\mu^*}} \omega
\end{aligned}$$

where $a_1 \neq 0$, a_2 and p_0 are arbitrary constants. The streamlines are shown in Figure 4.8.

Interchanging the roles of r and θ , we can study the family of curves $\frac{r - k(\theta)}{m(\theta)} = \text{constant}$ in the same manner with minor changes using the (θ, ψ) -net. As an example, we study the family of curves $re^{-m\theta} = \text{constant}$.

4.3.10. Example X. (Flow with $re^{-m\theta} = \text{constant}$ as streamlines).

We assume that

$$re^{-m\theta} = \gamma(\psi); \quad m \neq 0, \quad \gamma'(\psi) \neq 0 \quad (4.71)$$

where $\gamma(\psi)$ is an unknown function of ψ .

Using $x = r \cos\theta$ and $y = r \sin\theta$ in equations (2.24) and (2.26), we obtain

$$\begin{aligned}
E &= r^2 + \left(\frac{\partial r}{\partial \theta}\right)^2, \quad F = \frac{\partial r}{\partial \theta} \frac{\partial r}{\partial \psi} \\
G &= \left(\frac{\partial r}{\partial \psi}\right)^2, \quad W = \sqrt{EG - F^2} = r \frac{\partial r}{\partial \psi}
\end{aligned} \quad (4.72)$$

and

$$J = \frac{\partial x}{\partial \theta} \frac{\partial y}{\partial \psi} - \frac{\partial x}{\partial \psi} \frac{\partial y}{\partial \theta} = -r \frac{\partial r}{\partial \psi} \quad (4.73)$$

Employing (4.71) in ((4.72) and (4.73), we get

$$\begin{aligned} E &= (1 + m^2) e^{2m\theta} \gamma^2(\psi), & F &= m e^{2m\theta} \gamma(\psi) \gamma'(\psi) \\ G &= e^{2m\theta} \gamma'^2(\psi), & J = -W &= -e^{2m\theta} \gamma(\psi) \gamma'(\psi) \end{aligned} \quad (4.74)$$

Substituting (4.74) in equations (3.14) to (3.15) with $\phi = \theta$ and J replaced by $-J$, we find that the Gauss equation (3.16) is identically satisfied and the other two equations give

$$\begin{aligned} \omega &= -(1 + m^2) e^{-2m\theta} \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' \\ \Omega &= \beta(\psi) \omega - (1 + m^2) e^{-2m\theta} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \end{aligned} \quad (4.75)$$

Using (4.74) and (4.75) in equation (3.17) with $\phi = \theta$ and J replaced by $-J$, we obtain

$$\begin{aligned} &-4m^2 (1 + m^2) \frac{1}{\gamma^2(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' - 4m^2 (1 + m^2) \left[\frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' \right]' \\ &- (1 + m^2)^2 \left\{ \frac{\gamma(\psi)}{\gamma'(\psi)} \left[\frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' \right]' \right\}' - 2m (1 + m^2) \frac{\mu^*}{\mu} \frac{\beta(\psi) \beta'(\psi)}{\gamma'^2(\psi)} \\ &+ 2m (1 + m^2) \frac{1}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' = 0 \end{aligned} \quad (4.76)$$

Equation (4.76) is one equation in two unknown functions $\beta(\psi)$ and $\gamma(\psi)$. There are two ways to find solutions of this equation. One way is to assume a form of $\gamma(\psi)$ and use equation (4.76) to find $\beta(\psi)$ and the other way is to assume a form of $\beta(\psi)$ and use equation (4.76) to obtain $\gamma(\psi)$.

The exact solutions of this flow are given by

$$\begin{aligned}
v_1 &= \frac{1}{r} \frac{\partial \psi}{\partial \theta} = -\frac{m \gamma(\psi)}{r \gamma'(\psi)} \\
v_2 &= -\frac{\partial \psi}{\partial r} = -\frac{\gamma(\psi)}{r \gamma'(\psi)} \\
H_1 &= \frac{1}{r} [\sin \theta - m \cos \theta] \frac{\gamma(\psi) \beta(\psi)}{\gamma'(\psi)} \\
H_2 &= -\frac{1}{r} [\cos \theta + m \sin \theta] \frac{\gamma(\psi) \beta(\psi)}{\gamma'(\psi)} \\
p &= \frac{(1+m^2)}{2r^2} \left\{ 2m\mu \frac{\gamma(\psi)}{\gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' - \frac{\rho \gamma^2(\psi)}{\gamma'^2(\psi)} \right. \\
&\quad \left. + \frac{\mu(1+m^2)}{m} \frac{\gamma^3(\psi)}{\gamma'(\psi)} \left[\frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi)}{\gamma'(\psi)} \right)' \right]' \right\} + p_0
\end{aligned}$$

where p_0 is an arbitrary constant and $\gamma(\psi)$, $\beta(\psi)$ are arbitrary functions of ψ such that equation (4.76) is satisfied. The function $\gamma(\psi)$ must be chosen such that the pressure function is single-valued. Figure 4.9. shows the streamline pattern for this rotational flow.

$$\theta - m_1 r^3 - m_2 r^2 = c, \quad m_1 = 1, \quad m_2 = 1 \quad dr = 0.1$$

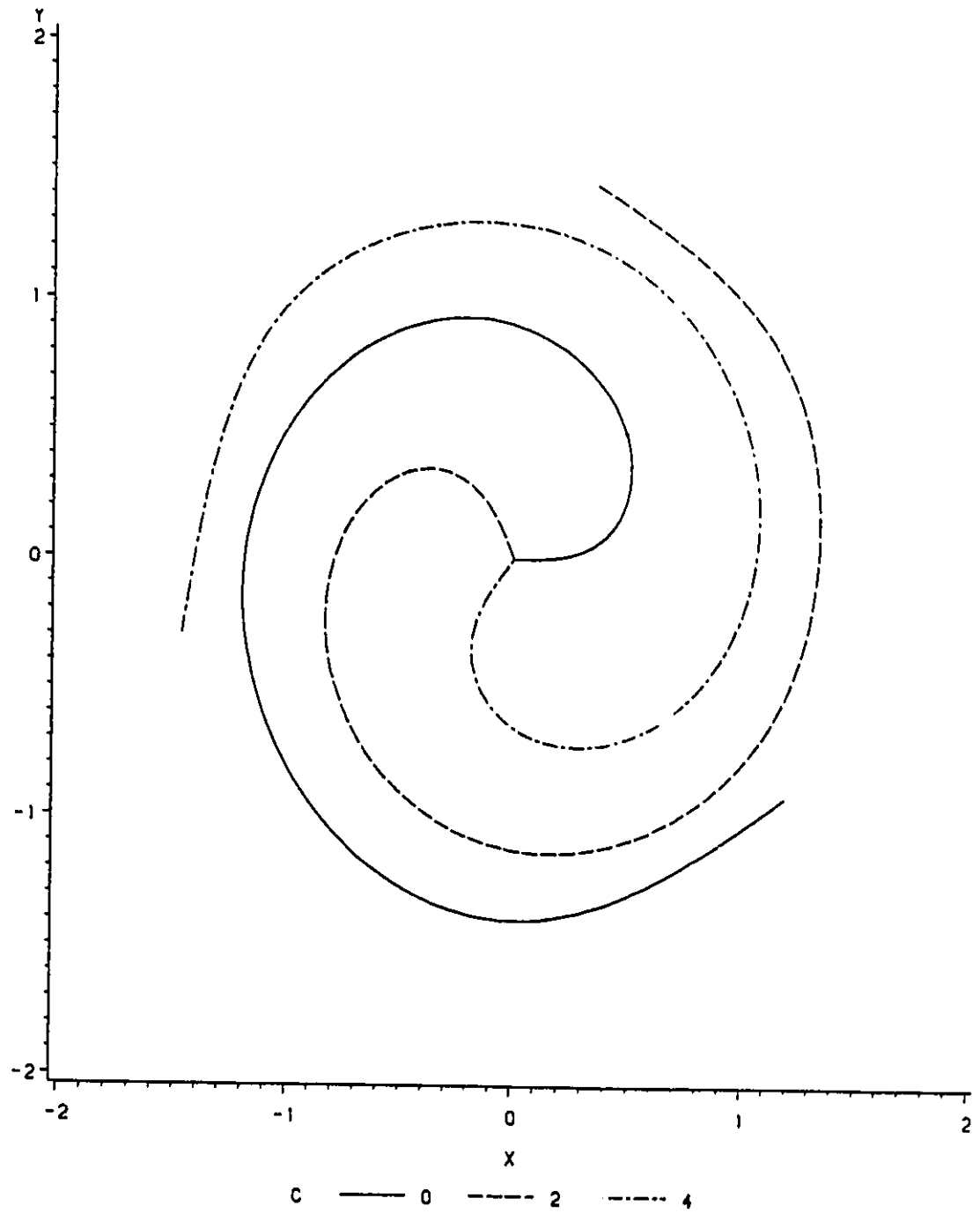


FIGURE 4.1: Streamline pattern for $\theta - m_1 r^3 - m_2 r^2 = \text{constant}$.

$$\text{Theta} - ar = c, a=1$$

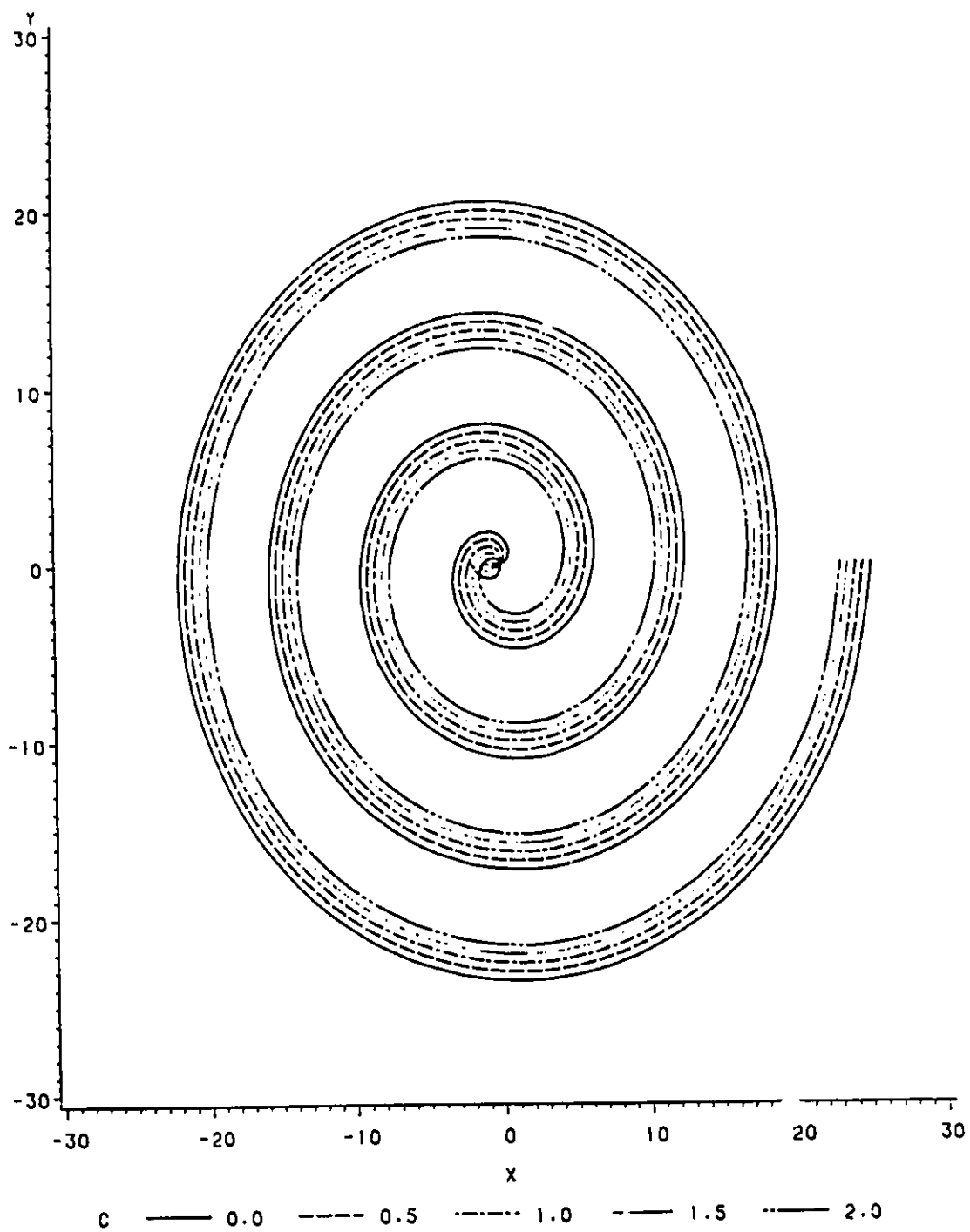


FIGURE 4.2: Streamline pattern for $\theta - ar = \text{constant}$.

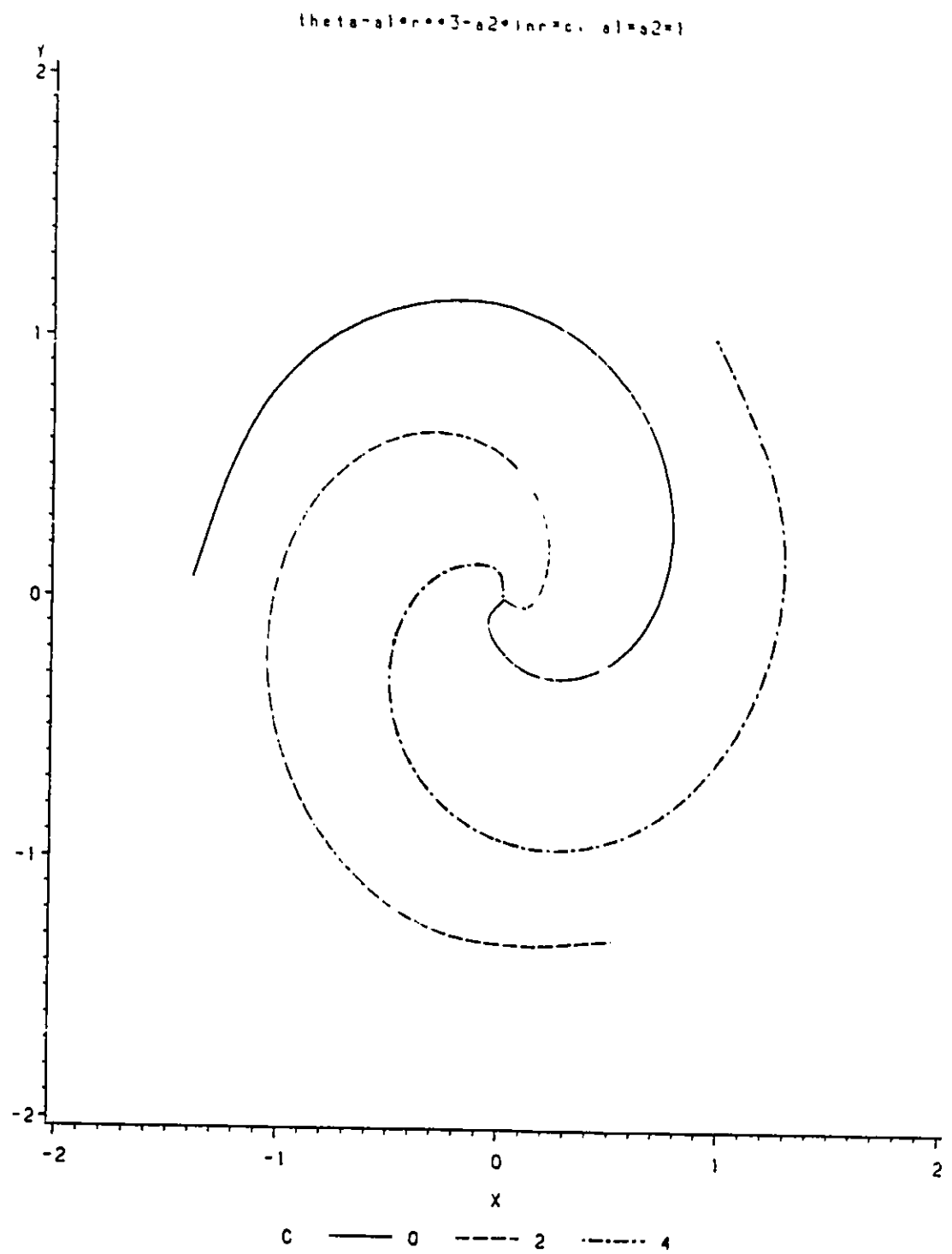


FIGURE 4.3: Streamline pattern for $\theta - a_1 r^3 - a_2 \ln r = \text{constant}$.

$$\theta - \ln r = c/r^2$$

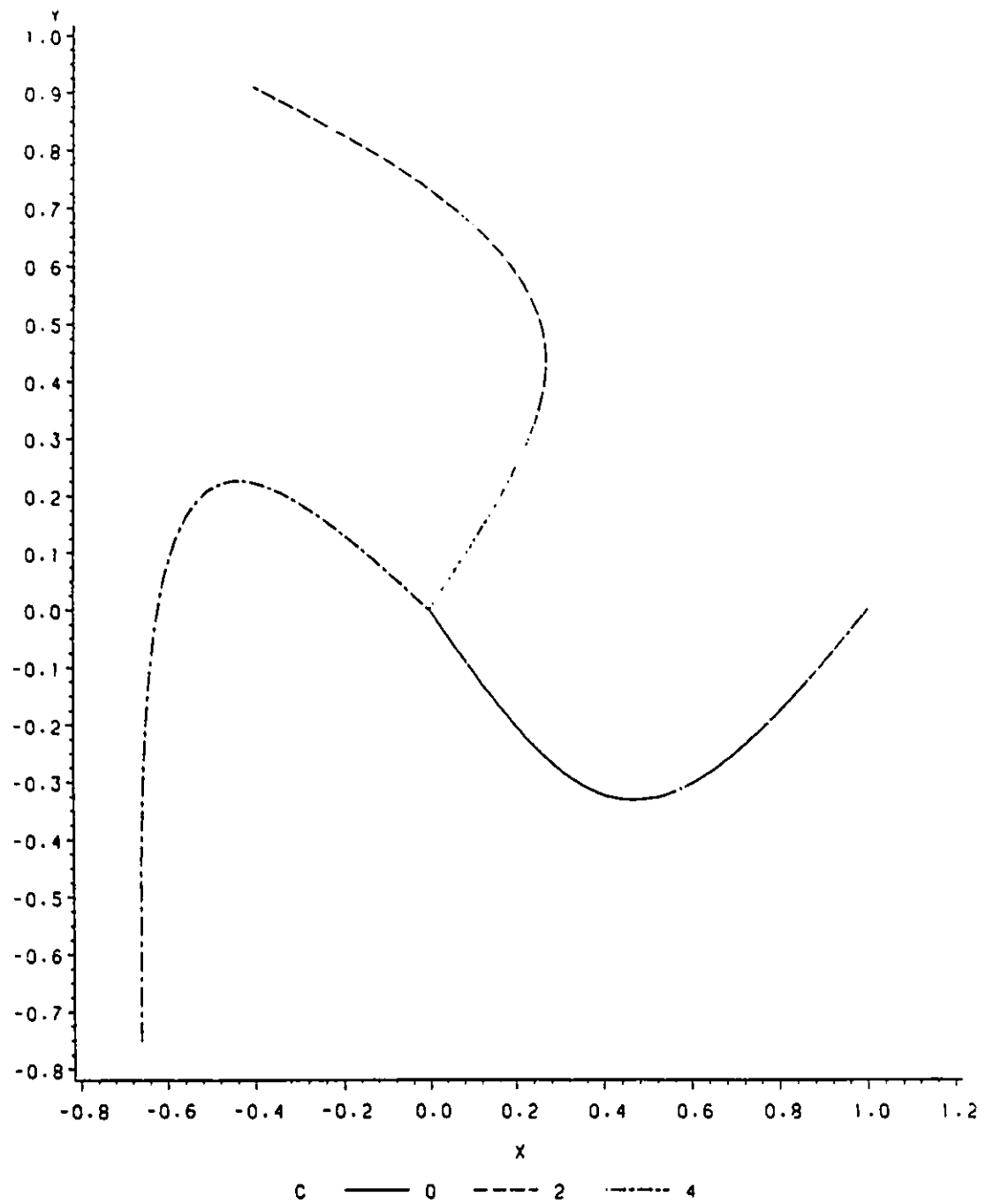


FIGURE 4.4: Streamline pattern for $r^2(\theta - \ln r) = \text{constant}$.

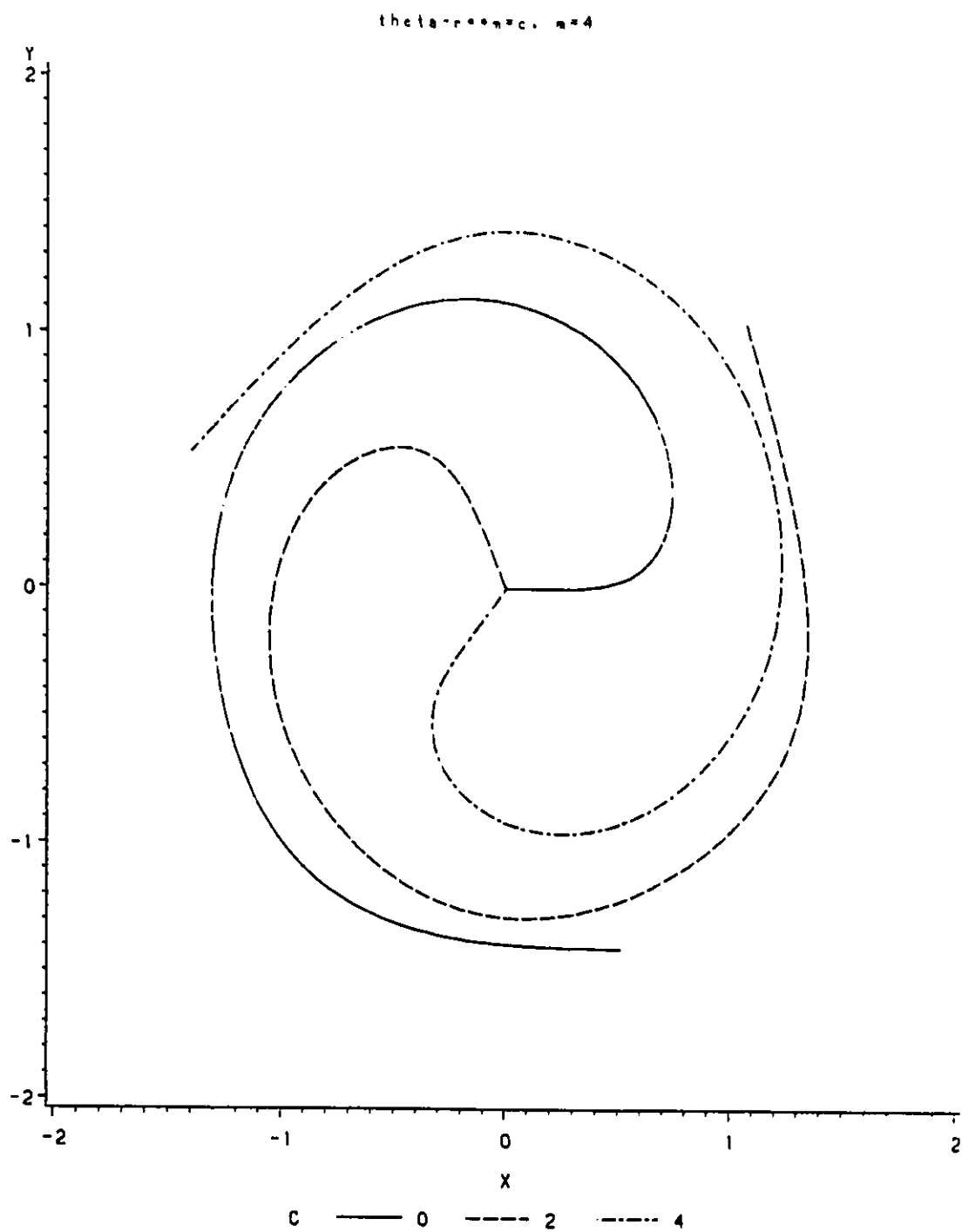


FIGURE 4.5: Streamline pattern for $\theta - r^m = \text{constant}$.

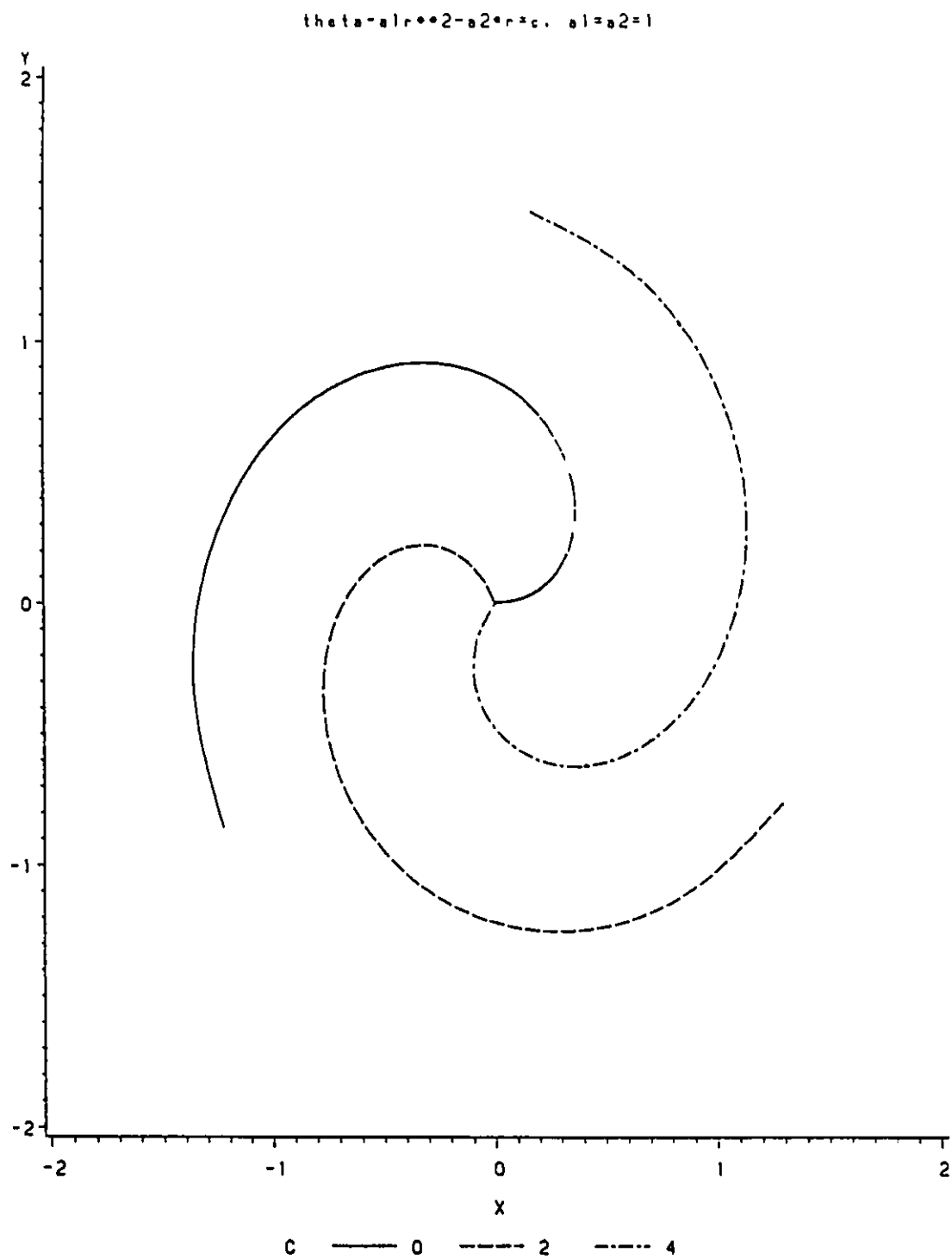


FIGURE 4.6: Streamline pattern for $\theta - a_1 r^2 - a_2 r = \text{constant}$.

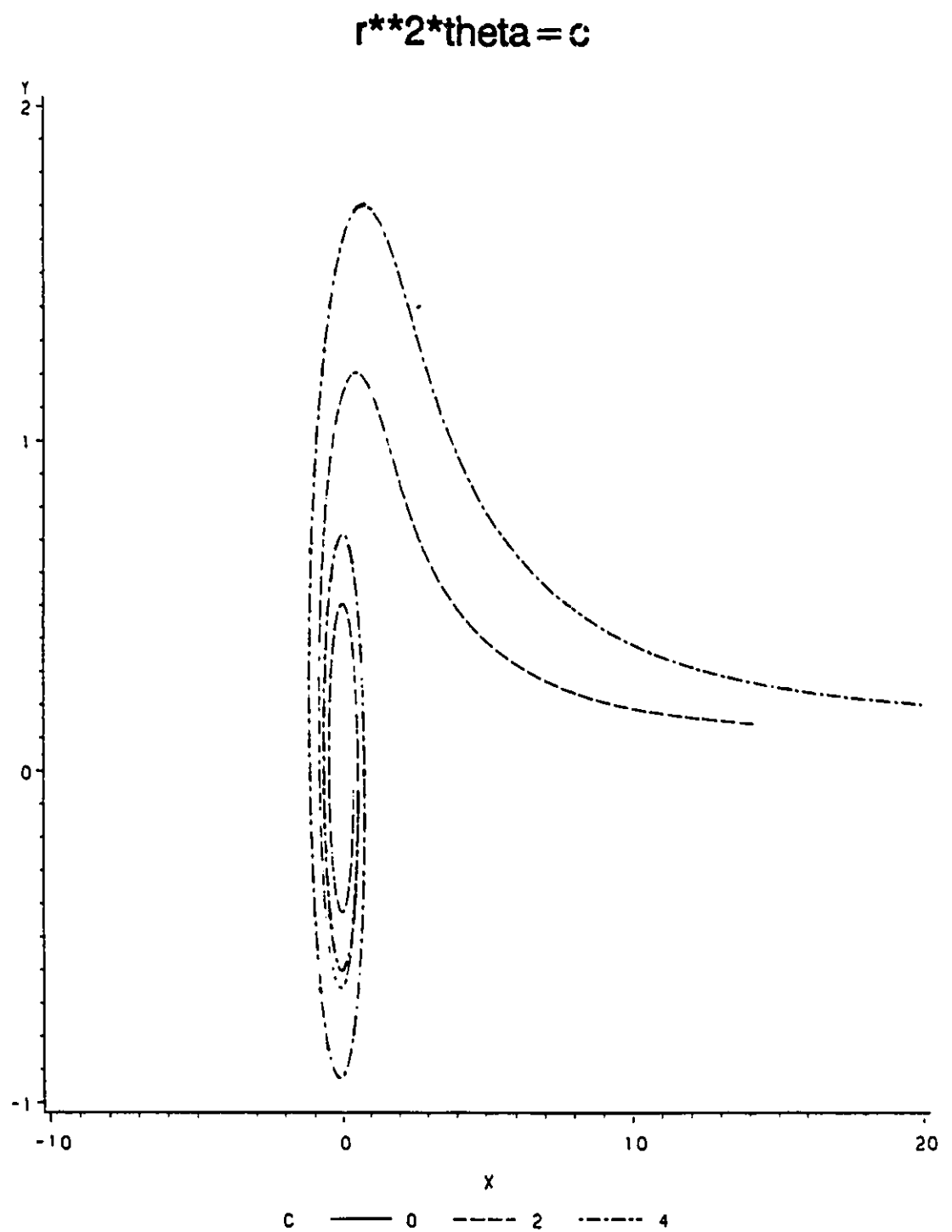


FIGURE 4.8: Streamline pattern for $r^2\theta = \text{constant}$.

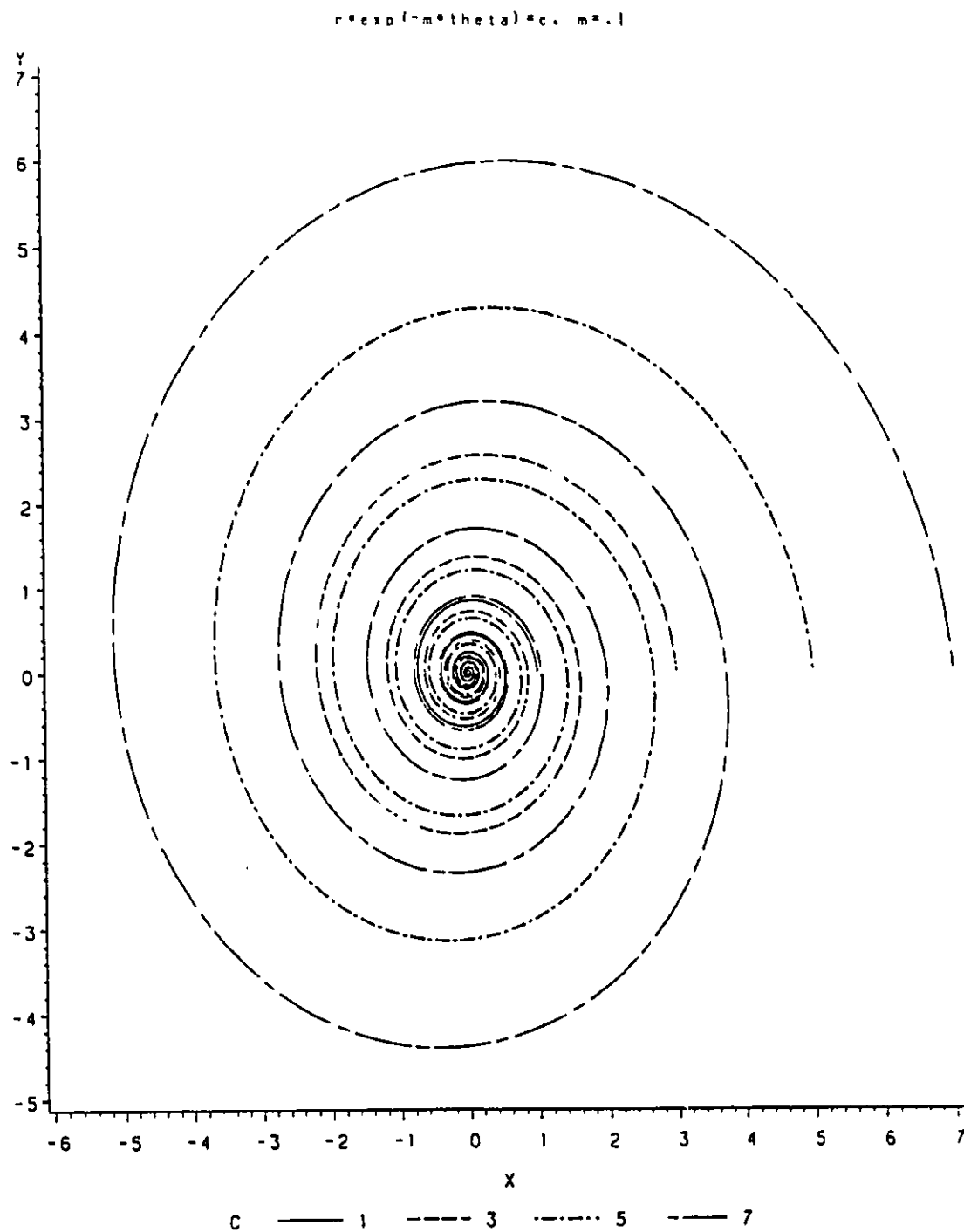


FIGURE 4.9: Streamline pattern for $re^{-m\theta} = \text{constant}$.

CHAPTER 5

EXACT SOLUTIONS OF STEADY PLANE FLOWS USING (ξ, ψ) - OR (η, ψ) -COORDINATES

5.1 INTRODUCTION.

This chapter deals with flows when the streamline patterns are of the form

$$\frac{\eta - f(\xi)}{g(\xi)} = \text{constant}$$

and

$$\frac{\xi - k(\eta)}{m(\eta)} = \text{constant}$$

where $\xi(x, y) = \text{Re}[N(z)]$, $\eta(x, y) = \text{Im}[N(z)]$ and $N(z)$ is an analytic function of z . In the cases when $f(\xi) = 0$ and $g(\xi) = 1$ or $k(\eta) = 0$ and $m(\eta) = 1$, the problem is called an isometric flow problem or Hamel's problem and was first raised by Jeffery [1915]. However, Hamel [1916] was the first to give complete solutions of the permissible flow patterns for ordinary viscous incompressible plane flows so that

$$\xi(x, y) = \text{constant}$$

are the streamlines when $\xi(x, y) = \text{Re}[N(z)]$ or $\eta(x, y) = \text{Im}[N(z)]$ and $N(z)$ is an analytic function of z .

Martin [1971] re-worked this problem for the study of ordinary fluids to illustrate his new approach. Prim [1952], Garg and Chandna [1976], Barron, Chandna and Prabakaran [1984] and Chandna and Chandna [1989] investigated Hamel's problem for rotational gas dynamics, for infinitely conducting orthogonal magnetohydrodynamics, for orthogonal electromagnetohydrodynamics and orthogonal electromagnetohydrodynamics respectively. Kaloni and Huschilt [1984] and Kaloni and Siddiqui [1983] examined this problem within the context of a second-grade fluid.

In our work, we give a method of solving flow problems for steady plane infinitely conducting MHD aligned flows when $N(z) = \xi(x, y) + i\eta(x, y)$, $f(\xi)$, $g(\xi)$, $k(\eta)$ and $m(\eta)$ are known functions. As examples to illustrate the method, we use two analytic functions $N(z) = \sqrt{2z}$ and $N(z) = \ln z$, and study the flows with the following streamline patterns:

- I. $\eta(x, y) - \xi(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- II. $\xi(x, y) - \eta^3(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- III. $\eta(x, y) - \xi(x, y) - \xi^3(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- IV. $\frac{\eta(x, y)}{\xi(x, y)} = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- V. $\eta(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- VI. $\xi(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- VII. $\eta(x, y) - \xi^3(x, y) = \text{constant}$ when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$,
- VIII. $\eta(x, y) = \xi(x, y) - \xi^2(x, y) - e^{2\xi(x, y)} = \text{constant}$ when $\ln z = \xi(x, y) + i\eta(x, y)$,
- IX. $\eta(x, y) - \xi^2(x, y) = \text{constant}$ when $\ln z = \xi(x, y) + i\eta(x, y)$,
- X. $\eta(x, y) = \text{constant}$ when $\ln z = \xi(x, y) + i\eta(x, y)$,
- XI. $\xi(x, y) = \text{constant}$ when $\ln z = \xi(x, y) + i\eta(x, y)$.

Examples V, VI, X and XI are four streamline patterns for the Hamel's problem for our flows. Two of these flow patterns are different from the four known flow patterns for Hamel's problem in ordinary viscous fluid dynamics.

5.2 FLOW EQUATIONS.

The steady plane flow of a viscous incompressible and electrically conducting fluid, in the presence of a magnetic field, is governed by equations (2.10) to (2.18).

Considering the flow to be aligned, we take

$$H_1 = \beta u, \quad H_2 = \beta v, \quad H_3 \equiv 0 \quad (5.1)$$

where $\beta(x, y)$ is an arbitrary function constant on each individual streamline.

Proceeding as in sections 3.2 and 3.3, our flow in (ϕ, ψ) -net is governed by:

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \quad (5.2)$$

$$\Omega = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} \quad (5.3)$$

$$\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) = 0 \quad (5.4)$$

$$\mu \left[\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial \omega}{\partial \phi} - \frac{F}{J} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} - \frac{F}{J} \frac{\partial \omega}{\partial \phi} \right) \right] - \rho \frac{\partial \omega}{\partial \phi} + \mu^* \beta \frac{\partial \Omega}{\partial \phi} = 0 \quad (5.5)$$

$$\beta = \beta(\psi) \quad (5.6)$$

This is a system of five equations in six unknowns E, F, G, ω, Ω and β as functions of ϕ, ψ .

5.3. METHOD.

Let $w = \xi + i\eta$ be an analytic function of $z = x + iy$ where $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$. Since w is an analytic function of x, y , then its real and imaginary parts must satisfy the Cauchy-Riemann equations, that is

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y}, \quad \frac{\partial \xi}{\partial y} = -\frac{\partial \eta}{\partial x} \quad (5.7)$$

The equations $\xi = \xi(x, y)$ and $\eta = \eta(x, y)$ can be solved to obtain

$$x = x(\xi, \eta), \quad y = y(\xi, \eta) \quad (5.8)$$

such that

$$\frac{\partial x}{\partial \xi} = J^* \frac{\partial \eta}{\partial y}, \quad \frac{\partial x}{\partial \eta} = -J^* \frac{\partial \xi}{\partial y}, \quad \frac{\partial y}{\partial \xi} = -J^* \frac{\partial \eta}{\partial x}, \quad \frac{\partial y}{\partial \eta} = J^* \frac{\partial \xi}{\partial x} \quad (5.9)$$

provided $0 < |J^*| < \infty$, where J^* is the transformation Jacobian and is given by

$$J^* = \frac{\partial(x, y)}{\partial(\xi, \eta)} = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta} \frac{\partial y}{\partial \xi} \quad (5.10)$$

Using (5.9) and (5.7) in (5.10), we obtain

$$J^* = J^* \frac{\partial x}{\partial \xi} \frac{\partial \eta}{\partial y} - J^* \frac{\partial y}{\partial \xi} \frac{\partial \eta}{\partial x}$$

Employing (5.9) in this equation, we get

$$J^* = \left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \quad (5.11)$$

Similarly, we can show that

$$J^* = \left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \quad (5.12)$$

The square element of arc length in (x, y) -net is given by

$$ds^2 = dx^2 + dy^2 \quad (5.13)$$

Using (5.8), equation (5.13) gives

$$ds^2 = \left[\left(\frac{\partial x}{\partial \xi} \right)^2 + \left(\frac{\partial y}{\partial \xi} \right)^2 \right] d\xi^2 + 2 \left[\frac{\partial x}{\partial \xi} \frac{\partial x}{\partial \eta} + \frac{\partial y}{\partial \xi} \frac{\partial y}{\partial \eta} \right] d\xi d\eta + \left[\left(\frac{\partial x}{\partial \eta} \right)^2 + \left(\frac{\partial y}{\partial \eta} \right)^2 \right] d\eta^2 \quad (5.14)$$

which upon substitution of (5.9) and (5.7) implies

$$ds^2 = J^* [d\xi^2 + d\eta^2] \quad (5.15)$$

5.3.1 Method for the (ξ, ψ) -coordinate net.

To analyze whether a given family of curves $\frac{\eta - f(\xi)}{g(\xi)} = \text{constant}$ can or cannot be streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{\eta - f(\xi)}{g(\xi)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (5.16)$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$ and we take the coordinates lines $\phi = \text{constant}$ to be $\xi = \text{constant}$.

Employing equation (5.16) in (5.15) and simplifying the resulting equation, we obtain

$$ds^2 = J^* \left[1 + (f'(\xi) + g'(\xi)\gamma(\psi))^2 \right] d\xi^2 + 2J^* [f'(\xi) + g'(\xi)\gamma(\psi)] g(\xi) \gamma'(\psi) d\xi d\psi + J^* g^2(\xi) \gamma'^2(\psi) d\psi^2 \quad (5.17)$$

Comparing (5.17) with (2.24) after taking $\phi = \xi$, we get

$$\begin{aligned} E &= J^* \left\{ 1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2 \right\} \\ F &= J^* [f'(\xi) + g'(\xi)\gamma(\psi)] g(\xi) \gamma'(\psi) \\ G &= J^* g^2(\xi) \gamma'^2(\psi) \\ W &= \sqrt{EG - F^2} = J^* g(\xi) \gamma'(\psi) \end{aligned} \quad (5.18)$$

Since

$$J = \frac{\partial(x, y)}{\partial(\phi, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(\xi, \psi)},$$

then

$$J = J^* g(\xi) \gamma'(\psi)$$

and therefore

$$J = W = J^* g(\xi) \gamma'(\psi) \quad (5.19)$$

Equations (5.18) yields $E = J^* + \frac{F^2}{G}$. Therefore, the system of equations (5.2) to (5.6) becomes a determinate system of five equations in five unknowns F , G , ω , Ω and β .

Using (5.18), (5.19) and $\phi = \xi$ in (5.2) to (5.6), we have the following theorem:

Theorem 5.1. *If a steady, plane, viscous incompressible fluid of infinite electrical conductivity flows along $\frac{\eta - f(\xi)}{g(\xi)} = \text{constant}$ in the presence of an aligned magnetic field, then the known functions $f(\xi)$, $g(\xi)$ and the unknown functions $\beta(\psi)$ and $\gamma(\psi)$ must satisfy*

$$\begin{aligned} g(\xi) \gamma'(\psi) \frac{\partial^2 \omega}{\partial \xi^2} - 2[f'(\xi) + g'(\xi) \gamma(\psi)] \frac{\partial^2 \omega}{\partial \xi \partial \psi} + \left[\frac{1 + f'^2(\xi)}{g(\xi)} + \frac{2f'(\xi)g'(\xi)}{g(\xi)} \gamma(\psi) \right. \\ \left. + \frac{g'^2(\xi)}{g(\xi)} \gamma^2(\psi) \right] \frac{1}{\gamma'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} + \left\{ -f''(\xi) + \frac{2f'(\xi)g'(\xi)}{g(\xi)} + \left[\frac{2g'^2(\xi)}{g(\xi)} - g''(\xi) \right] \gamma(\psi) \right. \\ \left. - \frac{1 + f'^2(\xi)}{g(\xi)} \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2f'(\xi)g'(\xi)}{g(\xi)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{g'^2(\xi)}{g(\xi)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right\} \frac{\partial \omega}{\partial \psi} \\ - \frac{\rho}{\mu} \frac{\partial \omega}{\partial \xi} + \frac{\mu^*}{\mu} \beta(\psi) \frac{\partial \Omega}{\partial \xi} = 0 \end{aligned} \quad (5.20)$$

and

$$\begin{aligned} 2[f'(\xi) + g'(\xi) \gamma(\psi)] \frac{\partial^2 J^*}{\partial \xi \partial \psi} - \frac{1}{g(\xi) \gamma'(\psi)} \left\{ 1 + [f'(\xi) + g'(\xi) \gamma(\psi)]^2 \right\} \frac{\partial^2 J^*}{\partial \psi^2} \\ - g(\xi) \gamma'(\psi) \frac{\partial^2 J^*}{\partial \xi^2} + \left\{ \left[1 + [f'(\xi) + g'(\xi) \gamma(\psi)]^2 \right] \frac{\gamma''(\psi)}{g(\xi) \gamma'^2(\psi)} \right. \\ \left. + [f''(\xi) + g''(\xi) \gamma(\psi)] - 2 \frac{g'(\xi)}{g(\xi)} [f'(\xi) + g'(\xi) \gamma(\psi)] \right\} \frac{\partial J^*}{\partial \psi} \\ + \frac{1}{J^* g(\xi) \gamma'(\psi)} \left\{ 1 + [f'(\xi) + g'(\xi) \gamma(\psi)]^2 \right\} \left(\frac{\partial J^*}{\partial \psi} \right)^2 \\ + \frac{1}{J^*} g(\xi) \gamma'(\psi) \left(\frac{\partial J^*}{\partial \xi} \right)^2 - \frac{2}{J^*} [f'(\xi) + g'(\xi) \gamma(\psi)] \frac{\partial J^*}{\partial \psi} \frac{\partial J^*}{\partial \xi} = 0 \end{aligned} \quad (5.21)$$

where ω and Ω are given by

$$\omega = \frac{1}{J^*} \left\{ \left[\frac{f''(\xi)}{g(\xi)} - \frac{2f'(\xi)g'(\xi)}{g^2(\xi)} \right] \frac{1}{\gamma'(\psi)} + \left[\frac{g''(\xi)}{g(\xi)} - \frac{2g'(\xi)}{g^2(\xi)} \right] \frac{\gamma(\psi)}{\gamma'(\psi)} \right. \\ \left. + \frac{1+f'^2(\xi)}{g^2(\xi)} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2f'(\xi)g'(\xi)}{g^2(\xi)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{g'^2(\xi)}{g^2(\xi)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} \right\} \quad (5.22)$$

$$\Omega = \beta(\psi)\omega - \frac{1}{J^*g^2(\xi)} \left\{ 1 + [f'(\xi) + g'(\xi)\gamma(\psi)]^2 \right\} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \quad (5.23)$$

and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

A given family of curves $\frac{\eta - f(\xi)}{g(\xi)} = \text{constant}$ is a permissible family of streamlines if and only if the solution obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$.

5.3.2 Method for the (η, ψ) -coordinate net.

To analyze whether a given family of curves $\frac{\xi - k(\eta)}{m(\eta)} = \text{constant}$ can or cannot be streamlines, we assume the affirmative so that there exists some function $\gamma(\psi)$ such that

$$\frac{\xi - k(\eta)}{m(\eta)} = \gamma(\psi), \quad \gamma'(\psi) \neq 0 \quad (5.24)$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$ and we take the coordinates lines $\phi = \text{constant}$ to be $\eta = \text{constant}$.

Employing equation (5.24) in (5.15) and simplifying the resulting equation, we obtain

$$ds^2 = J^* \left[1 + (k'(\eta) + m'(\eta)\gamma(\psi))^2 \right] d\eta^2 \\ + 2J^* [k'(\eta) + m'(\eta)\gamma(\psi)] m(\eta) \gamma'(\psi) d\eta d\psi + J^* m^2(\eta) \gamma'^2(\psi) d\psi^2 \quad (5.25)$$

Comparing (5.25) with (2.24) after taking $\phi = \xi$, we get

$$E = J^* \left\{ 1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2 \right\} \\ F = J^* [k'(\eta) + m'(\eta)\gamma(\psi)] m(\eta) \gamma'(\psi) \\ G = J^* m^2(\eta) \gamma'^2(\psi) \\ W = \sqrt{EG - F^2} = J^* m(\eta) \gamma'(\psi) \quad (5.26)$$

Since

$$J = \frac{\partial(x, y)}{\partial(\phi, \psi)} = \frac{\partial(x, y)}{\partial(\eta, \psi)} = \frac{\partial(x, y)}{\partial(\xi, \eta)} \frac{\partial(\xi, \eta)}{\partial(\eta, \psi)},$$

then

$$J = -J^* m(\eta) \gamma'(\psi)$$

and therefore

$$J = -W = -J^* m(\eta) \gamma'(\psi) \quad (5.27)$$

Equation (5.26) yields $E = J^* + \frac{F^2}{G}$. Therefore, the system of equations (5.2) to (5.6) becomes a determinate system of five equations in five unknowns F , G , ω , Ω and β .

Using (5.26), (5.27) and $\phi = \eta$ in (5.2) to (5.6), we have the following theorem:

Theorem 5.2. *If a steady, plane, viscous, incompressible fluid of infinite electrical conductivity flows along $\frac{\xi - k(\eta)}{m(\eta)} = \text{constant}$ in the presence of aligned magnetic field, then the known functions $k(\eta)$, $m(\eta)$ and the unknown functions $\beta(\psi)$, $\gamma(\psi)$ must satisfy*

$$\begin{aligned} m(\eta) \gamma'(\psi) \frac{\partial^2 \omega}{\partial \eta^2} - 2 [k'(\eta) + m'(\eta) \gamma(\psi)] \frac{\partial^2 \omega}{\partial \eta \partial \psi} + \left[\frac{1 + k'^2(\eta)}{m(\eta)} + \frac{2k'(\eta)m'(\eta)}{m(\eta)} \gamma(\psi) \right. \\ \left. + \frac{m'^2(\eta)}{m(\eta)} \gamma^2(\psi) \right] \frac{1}{\gamma'(\psi)} \frac{\partial^2 \omega}{\partial \psi^2} + \left\{ -k''(\eta) + \frac{2k'(\eta)m'(\eta)}{m(\eta)} + \left[\frac{2m'^2(\eta)}{m(\eta)} - m''(\eta) \right] \gamma(\psi) \right. \\ \left. - \frac{1 + k'^2(\eta)}{m(\eta)} \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{2k'(\eta)m'(\eta)}{m(\eta)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{m'^2(\eta)}{m(\eta)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right\} \frac{\partial \omega}{\partial \psi} \\ + \frac{\rho}{\mu} \frac{\partial \omega}{\partial \eta} - \frac{\mu^*}{\mu} \beta(\psi) \frac{\partial \Omega}{\partial \eta} = 0 \end{aligned} \quad (5.28)$$

and

$$\begin{aligned}
& 2[k'(\eta) + m'(\eta)\gamma(\psi)] \frac{\partial^2 J^*}{\partial \eta \partial \psi} - \frac{1}{m(\eta)\gamma'(\psi)} \left\{ 1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2 \right\} \frac{\partial^2 J^*}{\partial \psi^2} \\
& - m(\eta)\gamma'(\psi) \frac{\partial^2 J^*}{\partial \eta^2} + \left\{ \left[1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2 \right] \frac{\gamma''(\psi)}{m(\eta)\gamma'^2(\psi)} \right. \\
& + [k''(\eta) + m''(\eta)\gamma(\psi)] - 2 \frac{m'(\eta)}{m(\eta)} [k'(\eta) + m'(\eta)\gamma(\psi)] \left. \right\} \frac{\partial J^*}{\partial \psi} \\
& + \frac{1}{J^* m(\eta)\gamma'(\psi)} \left\{ 1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2 \right\} \left(\frac{\partial J^*}{\partial \psi} \right)^2 \\
& + \frac{1}{J^*} m(\eta)\gamma'(\psi) \left(\frac{\partial J^*}{\partial \eta} \right)^2 - \frac{2}{J^*} [k'(\eta) + m'(\eta)\gamma(\psi)] \frac{\partial J^*}{\partial \psi} \frac{\partial J^*}{\partial \eta} = 0
\end{aligned} \tag{5.29}$$

where ω and Ω are given by

$$\begin{aligned}
\omega = \frac{1}{J^*} \left\{ \left[\frac{k''(\eta)}{m(\eta)} - \frac{2k'(\eta)m'(\eta)}{m^2(\eta)} \right] \frac{1}{\gamma'(\psi)} + \left[\frac{m''(\eta)}{m(\eta)} - \frac{2m'^2(\eta)}{m^2(\eta)} \right] \frac{\gamma(\psi)}{\gamma'(\psi)} \right. \\
\left. + \frac{1 + k'^2(\eta)}{m^2(\eta)} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{2k'(\eta)m'(\eta)}{m^2(\eta)} \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{m'^2(\eta)}{m^2(\eta)} \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^3(\psi)} \right\}
\end{aligned} \tag{5.30}$$

$$\Omega = \beta(\psi)\omega - \frac{1}{J^* m^2(\eta)} \left\{ 1 + [k'(\eta) + m'(\eta)\gamma(\psi)]^2 \right\} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \tag{5.31}$$

and $\gamma(\psi)$ is some function of ψ such that $\gamma'(\psi) \neq 0$.

Having established a streamline pattern, exact solutions for the flow pattern are then obtained.

5.4 APPLICATIONS.

We use analytic functions $w = \xi + i\eta = N(z) = \sqrt{2z}$ in the first seven examples and $w = \xi + i\eta = N(z) = \ln z$ in the other four examples.

5.4.1 Examples for $w = \sqrt{2z}$.

Let $z = \frac{1}{2}w^2$ or $w = \sqrt{2z}$. Then, we have

$$\left. \begin{aligned} x &= \frac{1}{2}(\xi^2 - \eta^2) \\ y &= \xi\eta \end{aligned} \right\} \quad (5.32)$$

or

$$\left. \begin{aligned} \xi &= \sqrt{x + \sqrt{x^2 + y^2}} \\ \eta &= \sqrt{-x + \sqrt{x^2 + y^2}} \end{aligned} \right\} \quad (5.33)$$

Using equation (5.32) in (5.11), we obtain

$$J^* = \xi^2 + \eta^2 \quad (5.34)$$

5.4.1.1 Example I. (Flow with $\eta - \xi = \text{constant}$ as streamlines).

We assume that

$$\eta - \xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.35)$$

where $\gamma(\psi)$ is an unknown function of ψ . Comparing (5.35) with (5.16), we get

$$f(\xi) = \xi, \quad g(\xi) = 1 \quad (5.36)$$

Using (5.35), equation (5.34) yields

$$J^* = 2\xi^2 + 2\xi\gamma(\psi) + \gamma^2(\psi) \quad (5.37)$$

Employing (5.22), (5.23), (5.36) and (5.37) in equations (5.20) and (5.21), we find that equation (5.21) is identically satisfied and (5.20) reduces to

$$\sum_{n=0}^4 A_n(\psi) \xi^n = 0 \quad (5.38)$$

where

$$\begin{aligned}
A_0(\psi) &= 4\gamma^4(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 8\gamma^3(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{8\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{4}{\mu}\gamma^3(\psi) \left\{ [\rho - \mu^*\beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_1(\psi) &= 16\gamma^3(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 16\gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{16\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{16}{\mu}\gamma^2(\psi) \left\{ [\rho - \mu^*\beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_2(\psi) &= 32\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 16\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{16\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{24}{\mu} \left\{ [\rho - \mu^*\beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_3(\psi) &= 32\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{16}{\mu} \left\{ [\rho - \mu^*\beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
A_4(\psi) &= 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'
\end{aligned}$$

Equation (5.38) is a fourth degree polynomial in ξ with coefficients as functions of ψ only. Since ξ, ψ are independent variables, it follows that equation (5.38) can hold true for all values of ξ if all the coefficients of this polynomial vanish simultaneously and we have

$$A_4(\psi) = A_3(\psi) = A_2(\psi) = A_1(\psi) = A_0(\psi) = 0 \quad (5.39)$$

Integrating $A_4(\psi) = 0$ four times with respect to ψ , we obtain

$$a_1\gamma^3(\psi) + a_2\gamma^2(\psi) + a_3\gamma(\psi) + a_4 - \psi = 0 \quad (5.40)$$

where a_1, a_2, a_3 and a_4 are arbitrary constants of integration that are not zero simultaneously.

Using $A_4(\psi) = 0$ in $A_3(\psi) = 0$, we get

$$\left\{ [\rho - \mu^*\beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} = 0$$

which upon integration implies that

$$\beta^2(\psi) = \frac{1}{\mu^*} [\rho - a_5 \gamma'^2(\psi)] \quad (5.41)$$

where a_5 is an arbitrary constant and $\gamma(\psi)$ is given implicitly by equation (5.40). Employing (5.40) and (5.41), $A_2(\psi) = 0$ gives

$$a_2 = 0 \quad (5.42)$$

Finally, using (5.39) to (5.42) in $A_1(\psi) = 0$ and $A_0(\psi) = 0$, we find that both of these equations are identically satisfied. Hence, the family of curves $\eta - \xi = \text{constant}$ are permissible streamlines for the flow under consideration and the unknown function $\gamma(\psi)$ is given implicitly by equation (5.40) with $a_2 = 0$.

Employing (5.35) in equation (5.40) with $a_2 = 0$, we get

$$\psi = a_1 (\eta - \xi)^3 + a_3 (\eta - \xi) + a_4 \quad (5.43)$$

where ξ and η as functions of x and y are given by equation (5.33). Thus, the solutions for the velocity components, the magnetic field components, the pressure, the vorticity and the current density are given by

$$\begin{aligned} u &= \frac{1}{2\sqrt{x^2 + y^2}} \left[6a_1 (\sqrt{x^2 + y^2} - y) + a_3 \right] \left\{ \sqrt{x + \sqrt{x^2 + y^2}} - \sqrt{-x + \sqrt{x^2 + y^2}} \right\} \\ v &= \frac{1}{2\sqrt{x^2 + y^2}} \left[6a_1 (\sqrt{x^2 + y^2} - y) + a_3 \right] \left\{ \sqrt{x + \sqrt{x^2 + y^2}} + \sqrt{-x + \sqrt{x^2 + y^2}} \right\} \\ H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\ p &= -6a_1\mu \frac{1}{\sqrt{x^2 + y^2}} \left[\sqrt{x + \sqrt{x^2 + y^2}} + \sqrt{-x + \sqrt{x^2 + y^2}} \right] \\ &\quad - \frac{\rho}{2} \frac{1}{\sqrt{x^2 + y^2}} \left[6a_1 (\sqrt{x^2 + y^2} - y) + a_3 \right]^2 + p_0 \\ \omega &= -\frac{1}{\sqrt{x^2 + y^2}} \left[6a_1 \sqrt{-x + \sqrt{x^2 + y^2}} - 6a_1 \sqrt{x + \sqrt{x^2 + y^2}} \right] \\ \Omega &= \beta(\psi)\omega - \frac{1}{\sqrt{x^2 + y^2}} \beta'(\psi) \left[6a_1 (\sqrt{x^2 + y^2} - y) + a_3 \right]^2 \end{aligned} \quad (5.44)$$

where p_0 is an arbitrary constant and $\beta(\psi)$ is given by equation (5.41). Summing up, we have the following theorem:

Theorem 5.3. *Streamline pattern $\eta - \xi = \text{constant}$ of steady plane motion is permissible for infinitely conducting MHD aligned flow and the solutions are given by equations (5.44).*

The streamline pattern for this flow is shown in Figure 5.1.

5.4.1.2. Example II. (Flow with $\xi - \eta^3 = \text{constant}$ as streamlines)

We let

$$\xi - \eta^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.45)$$

where $\gamma(\psi)$ is an unknown function of ψ and ξ, η are given by equations (5.33).

Upon substitution of (5.45), equation (5.34) yields

$$J^* = \eta^6 + 2\eta^3\gamma(\psi) + \eta^2 + \gamma^2(\psi) \quad (5.46)$$

Comparing (5.45) with (5.24), we get

$$k(\eta) = \eta^3, \quad m(\eta) = 1 \quad (5.47)$$

Employing (5.30), (5.31), (5.46) and (5.47) in equations (5.28) and (5.29), we find that equation (5.29) is identically satisfied and equation (5.28) reduces to

$$\sum_{n=0}^{20} B_n(\psi) \eta^n = 0 \quad (5.48)$$

where the coefficients $B_n(\psi)$ are given by

$$B_{20}(\psi) = 81 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$B_{19}(\psi) = B_{18}(\psi) = 0$$

$$B_{17}(\psi) = 324\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 648 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)'$$

$$B_{16}(\psi) = 180 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$B_{15}(\psi) = -\frac{18}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$B_{14}(\psi) = 486\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 2268 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 1760 \frac{\gamma''(\psi)}{\gamma'^2(\psi)}$$

$$B_{13}(\psi) = 396\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 1128 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)'$$

$$B_{12}(\psi) = 118 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + 180 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{30}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\ - \frac{18}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$B_{11}(\psi) = 324\gamma^3(\psi) - 2916\gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4692\gamma(\psi) \frac{\gamma''(\psi)}{\gamma'(\psi)} \\ + \frac{1}{\mu} [48 - 54\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$B_{10}(\psi) = 270\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 1992\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 2204 \frac{\gamma''(\psi)}{\gamma'^2(\psi)}$$

$$B_9(\psi) = 76\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 504 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + [720\gamma(\psi) + 48] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ - \frac{84}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma(\psi)}{\gamma'(\psi)} + \frac{108\gamma^2(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$\begin{aligned}
B_8(\psi) = & [20 + 81\gamma^4(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 1620\gamma^3(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [360 + 5400\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
& + \frac{36\gamma(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
B_7(\psi) = & 72\gamma^3(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 936\gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 1928 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
& + \frac{1}{\mu} [30 - 24\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
B_6(\psi) = & 44\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 108\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [348 + 48\gamma(\psi) + 1080\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{72\gamma(\psi)}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
& + \frac{144\gamma^3(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
B_5(\psi) = & 4\gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - [24 + 324\gamma^4(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [720\gamma(\psi) + 1172\gamma^3(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{36}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma(\psi)}{\gamma'(\psi)} \\
& + \frac{60\gamma^2(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
B_4(\psi) = & [18\gamma^4(\psi) + 1] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 84\gamma^3(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [360\gamma^2(\psi) + 180] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{6}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
& - \frac{1}{\mu} [4\gamma(\psi) + 6\gamma^3(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
B_3(\psi) &= 4\gamma^3(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 12\gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + [720\gamma^3(\psi) - 36\gamma(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - \frac{12}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma^3(\psi)}{\gamma'(\psi)} \\
&\quad + \frac{1}{\mu} [36\gamma^4(\psi) - 2\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
B_2(\psi) &= 2\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + [360\gamma^2(\psi) + 12] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
B_1(\psi) &= - [12\gamma^4(\psi) + 48\gamma^2(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - 12\gamma^3(\psi) \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
B_0(\psi) &= \gamma^4(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + [180\gamma^4(\psi) + 12\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\
&\quad + \frac{6}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma^4(\psi)}{\gamma'(\psi)}
\end{aligned}$$

Since equation (5.48) is to hold true for all values of η , it follows that all the coefficients $B_n(\psi)$, $n = 0, 1, \dots, 20$ must vanish simultaneously. This requirement yields

$$\gamma(\psi) = b_1\psi + b_2, \quad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}} \quad (5.49)$$

where $b_1 \neq 0$ and b_2 are arbitrary constants. The exact solutions of this infinitely conducting MHD aligned flow are given by

$$\begin{aligned}
u &= \frac{1}{2b_1\sqrt{x^2+y^2}} \sqrt{-x + \sqrt{x^2+y^2}} (1-3y) \\
v &= -\frac{1}{2b_1\sqrt{x^2+y^2}} \left[3 \left(\sqrt{-x + \sqrt{x^2+y^2}} \right)^3 + \sqrt{x + \sqrt{x^2+y^2}} \right] \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\
p &= \frac{1}{2b_1\sqrt{x^2+y^2}} \left\{ 6\mu \sqrt{x + \sqrt{x^2+y^2}} - \frac{\rho}{2b_1} \left[1 + 9 \left(-x + \sqrt{x^2+y^2} \right)^2 \right] \right\} \\
\omega &= \frac{3}{b_1} \frac{\sqrt{-x + \sqrt{x^2+y^2}}}{\sqrt{x^2+y^2}} \\
\Omega &= \sqrt{\frac{\rho}{\mu^*}} \omega
\end{aligned} \quad (5.50)$$

where p_0 is an arbitrary constant of integration. Thus, we have the following theorem:

Theorem 5.4. *Steady plane flow with $\xi - \eta^3 = \text{constant}$ as a family of streamlines is allowed by infinitely conducting MHD aligned flow with solutions given by equations (5.50).*

The flow pattern of this example is shown in Figure 5.2.

5.4.1.3 Example III. (Flow with $\eta - \xi - \xi^3 = \text{constant}$ as streamlines)

We assume that

$$\eta - \xi - \xi^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.51)$$

where $\gamma(\psi)$ is an unknown function of ψ and ξ, η are given by equations (5.33). Using (5.51) in equation (5.34), we obtain

$$J^* = \xi^6 + 2\xi^4 + 2\xi^3\gamma(\psi) + 2\xi^2 + 2\xi\gamma(\psi) + \gamma^2(\psi) \quad (5.52)$$

Comparing (5.51) with (5.16), we get

$$f(\xi) = \xi + \xi^3, \quad g(\xi) = 1 \quad (5.53)$$

Proceeding as in Example I, we find that equation (5.21) is identically satisfied and equation (5.20) takes the form

$$\sum_{n=0}^{17} C_n(\psi) \xi^n = 0 \quad (5.54)$$

where

$$C_{17}(\psi) = -342 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 81 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$C_{16}(\psi) = 162 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$\begin{aligned}
C_{15}(\psi) &= -1512 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 351 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{18}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
C_{14}(\psi) &= -1296 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + [540 + 243\gamma(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
C_{13}(\psi) &= -3540 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + [639 + 324\gamma(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{60}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
C_{12}(\psi) &= 60 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4752\gamma(\psi) + [819 + 810\gamma(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{18\gamma(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} + \frac{30}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
C_{11}(\psi) &= 108 [1 + \gamma(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [4608 + 1944\gamma^2(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + [645 + 756\gamma(\psi) + 243\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{84}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
C_{10}(\psi) &= 240 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 8076\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + [660 + 1107\gamma(\psi) + 162\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{6\gamma(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} + \frac{96}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
C_9(\psi) &= [288 + 528\gamma(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [3418 + 5184\gamma^2(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
&\quad + [400 + 720\gamma(\psi) + 567\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{84}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma(\psi)}{\gamma'(\psi)} \\
&\quad + \frac{1}{\mu} [64 - 54\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_8(\psi) = & [324 + 216\gamma(\psi) + 324\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [6480\gamma(\psi) + 1296\gamma^3(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [320 + 828\gamma(\psi) + 216\gamma^2(\psi) + 81\gamma^3(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& - \frac{72\gamma(\psi)}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} + \frac{144}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)}
\end{aligned}$$

$$\begin{aligned}
C_7(\psi) = & [276 + 900\gamma(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [1208 + 5580\gamma^2(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [156 + 384\gamma(\psi) + 540\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{180}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma(\psi)}{\gamma'(\psi)} \\
& + \frac{1}{\mu} [37 - 180\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_6(\psi) = & [128 + 360\gamma(\psi) + 900\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [2548\gamma(\psi) + 2160\gamma^3(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [88 + 372\gamma(\psi) + 144\gamma^2(\psi) + 108\gamma^3(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& - \frac{1}{\mu} [48\gamma(\psi) + 88\gamma^3(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
& + \frac{1}{\mu} [96 + 72\gamma^2(\psi)] [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)}
\end{aligned}$$

$$\begin{aligned}
C_5(\psi) = & [120 + 576\gamma(\psi) + 108\gamma^2(\psi) + 324\gamma^3(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \frac{144}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma(\psi)}{\gamma'(\psi)} \\
& - [164 + 2448\gamma^2(\psi) + 324\gamma^4(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [36 + 112\gamma(\psi) + 288\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& + \frac{1}{\mu} [48 + 180\gamma^2(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_4(\psi) = & [40 + 192\gamma(\psi) + 756\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [492\gamma(\psi) + 1092\gamma^3(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [12 + 96\gamma(\psi) + 48\gamma^2(\psi) + 72\gamma^3(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& + \frac{1}{\mu} [56\gamma(\psi) - 138\gamma^3(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} \\
& + \frac{1}{\mu} [24 + 72\gamma^2(\psi)] [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)}
\end{aligned}$$

$$\begin{aligned}
C_3(\psi) = & [24 + 136\gamma(\psi) + 72\gamma^2(\psi) + 456\gamma^3(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + [4 - 440\gamma^2(\psi) \\
& - 216\gamma^4(\psi)] \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \frac{1}{\mu} [24\gamma(\psi) + 12\gamma^2(\psi)] [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\
& + [4 + 16\gamma(\psi) + 84\gamma^2(\psi)] \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& + \frac{1}{\mu} [16 - 36\gamma^4(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_2(\psi) = & [16 + 48\gamma(\psi) + 144\gamma^2(\psi) + 108\gamma^4(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - [12 + 216\gamma^2(\psi)] \gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + [12 + 8\gamma(\psi) + 24\gamma^2(\psi)] \gamma(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& + \frac{1}{\mu} [24\gamma(\psi) - 24\gamma^3(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_1(\psi) = & [16 + 24\gamma(\psi) + 96\gamma^2(\psi)] \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - [16 + 48\gamma^2(\psi)] \gamma^2(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\
& + 12\gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - \frac{12}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma^3(\psi)}{\gamma'(\psi)} \\
& + \frac{1}{\mu} [16\gamma^2(\psi) - 12\gamma^4(\psi)] \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}
\end{aligned}$$

$$\begin{aligned}
C_0(\psi) = & 32 \frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} - 8\gamma^3(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4\gamma^3(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
& + \frac{4}{\mu} \gamma^3(\psi) \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\} - \frac{6}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma^4(\psi)}{\gamma'(\psi)}
\end{aligned}$$

The polynomial (5.54) in ξ , with coefficients as functions of ψ only, holds true for

all values of ξ provided that all coefficients vanish simultaneously. This requirement yields

$$\gamma(\psi) = c_1\psi + c_2, \quad \beta(\psi) = \sqrt{\frac{\rho}{\mu^*}} \quad (5.55)$$

where $c_1 \neq 0$ and c_2 are arbitrary constants. For this streamline pattern the exact solutions are given by

$$\begin{aligned} u &= \frac{1}{2c_1\sqrt{x^2+y^2}} \left\{ (1-3y) \sqrt{x+\sqrt{x^2+y^2}} - \sqrt{-x+\sqrt{x^2+y^2}} \right\} \\ v &= \frac{1}{2c_1\sqrt{x^2+y^2}} \left\{ \sqrt{-x+\sqrt{x^2+y^2}} + \sqrt{x+\sqrt{x^2+y^2}} + 3 \left(\sqrt{x+\sqrt{x^2+y^2}} \right)^3 \right\} \\ H_1 &= \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\ p &= -\frac{1}{2c_1\sqrt{x^2+y^2}} \left\{ 6\mu\sqrt{-x+\sqrt{x^2+y^2}} + \frac{\rho}{2c_1} \left[18x^2 + 9y^2 + 6x + 2 \right. \right. \\ &\quad \left. \left. + 6(3x+1)\sqrt{x^2+y^2} \right] \right\} + p_0 \\ \omega &= \frac{3}{c_1\sqrt{x^2+y^2}} \sqrt{x+\sqrt{x^2+y^2}} \\ \Omega &= \sqrt{\frac{\rho}{\mu^*}} \omega \end{aligned} \quad (5.56)$$

where p_0 is an arbitrary constant of integration. Thus, we have the following theorem:

Theorem 5.5. *Steady plane flow with $\eta - \xi - \xi^3 = \text{constant}$ as a family of streamlines is permissible for infinitely conducting MHD aligned flow with exact solutions given by (5.56).*

Figure 5.3 shows the streamline pattern of this flow.

5.4.1.4 Example IV. (Flow with $\frac{\eta}{\xi} = \text{constant}$ as streamlines).

We assume that

$$\eta = \xi \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.57)$$

where $\gamma(\psi)$ is an unknown function of ψ and ξ, η are given by equations (5.33).

Using (5.57) in equation (5.34), we obtain

$$J^* = \xi^2 [1 + \gamma^2(\psi)] \quad (5.58)$$

Comparing (5.56) with (5.16), we get

$$f(\xi) = 0, \quad g(\xi) = \xi \quad (5.59)$$

Proceeding as above, we find that equation (5.21) is identically satisfied and equation (5.20) reduces to

$$\begin{aligned} & -20\gamma'(\psi) \left\{ \frac{2\gamma(\psi)}{[1 + \gamma^2(\psi)]\gamma'(\psi)} - \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\} - \frac{1 + \gamma^2(\psi)}{\gamma'(\psi)} \left\{ \frac{2\gamma(\psi)}{[1 + \gamma^2(\psi)]\gamma'(\psi)} - \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}'' \\ & - \left[10\gamma(\psi) - \frac{1 + \gamma^2(\psi)}{\gamma'^2(\psi)}\gamma''(\psi) \right] \left\{ \frac{2\gamma(\psi)}{[1 + \gamma^2(\psi)]\gamma'(\psi)} - \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}' + \frac{4\mu^*}{\mu} \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\ & - \frac{4}{\mu} [\rho - \mu^*\beta^2(\psi)] \left\{ \frac{2\gamma(\psi)}{[1 + \gamma^2(\psi)]\gamma'(\psi)} - \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\} = 0 \end{aligned} \quad (5.60)$$

Assuming that $\gamma(\psi)$ is an arbitrary function of ψ , then equation (5.60) can be solved to obtain $\beta(\psi)$. Letting $\gamma(\psi) = \psi$, then equation (5.60) after one integration gives

$$\beta^2(\psi) = \frac{1}{\mu^*} \left\{ \rho + \mu \left[6 - \frac{3\psi^4 - \beta_0}{(1 + \psi^2)^2} \right] \right\} \quad (5.61)$$

where β_0 is an arbitrary constant of integration. The solutions for this streamline

pattern with the assumption that $\gamma(\psi) = \psi$ are given by

$$\begin{aligned}
u &= \frac{x}{\sqrt{x^2 + y^2} (x + \sqrt{x^2 + y^2})} \\
v &= \frac{1}{\sqrt{x^2 + y^2}} \frac{y}{x + \sqrt{x^2 + y^2}} \\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\
p &= -\frac{1}{2(x + \sqrt{x^2 + y^2})^2} \left\{ \rho + 2\mu - \frac{\mu x}{\sqrt{x^2 + y^2}} \right\} + p_0 \\
\omega &= -\frac{y}{(x + \sqrt{x^2 + y^2})^2 \sqrt{x^2 + y^2}} \\
\Omega &= \beta(\psi)\omega - \frac{1}{(x + \sqrt{x^2 + y^2})^2} \beta'(\psi)
\end{aligned} \tag{5.62}$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by equation (5.61). Summing up, we have:

Theorem 5.6. *Steady plane flow with $\frac{\eta}{\xi} = \text{constant}$ as a family of streamlines is permissible for infinitely conducting MHD aligned flow with exact solutions given by equations (5.62).*

The flow pattern is shown in Figure 5.4.

5.4.1.5 Example V. (Flow with $\eta = \text{constant}$ as streamlines) .

This example gives us a streamline pattern for Hamel's problem for infinitely conducting MHD aligned flows. The streamline pattern obtained is not one of the four well known patterns for ordinary viscous fluid flow. This pattern is given in Figure 5.5.

We let

$$\eta = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{5.63}$$

where $\gamma(\psi)$ is an unknown function of ψ . Employing (5.63) in (5.34), we get

$$J^* = \xi^2 + \gamma^2(\psi) \tag{5.64}$$

Comparing (5.63) with (5.16), we have

$$f(\xi) = 0, \quad g(\xi) = 1 \quad (5.65)$$

Following the same procedure as in previous examples, we find that equation (5.21) is identically satisfied and equation (5.20) gives

$$\sum_{n=0}^2 D_n(\psi) \xi^n = 0 \quad (5.66)$$

where

$$\begin{aligned} D_0(\psi) &= 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4\gamma(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \gamma^2(\psi) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right) \right]' \\ D_1(\psi) &= [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \\ D_2(\psi) &= \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right) \right]' \end{aligned}$$

Integrating $D_2(\psi) = 0$ four times with respect to ψ , we obtain

$$c_1 \gamma^3(\psi) + c_2 \gamma^2(\psi) + c_3 \gamma(\psi) + c_4 - \psi = 0 \quad (5.67)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants that are not zero simultaneously.

Using equation (5.67) in $D_0(\psi) = 0$, we get

$$c_2 = 0 \quad (5.68)$$

Employing equation (5.67) in $D_1(\psi) = 0$ and integrating the resulting equation once with respect to ψ , we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*} \left[\rho - \frac{c_3}{\left(3c_1 \sqrt{x^2 + y^2} - 3c_1 x + c_3 \right)^2} \right] \quad (5.69)$$

where c_5 is an arbitrary constant of integration. Substituting equation (5.63) in (5.67) with $c_2 = 0$, we find that

$$\psi = c_1 \eta^3 + c_3 \eta + c_4 \quad (5.70)$$

where η is given by equation (5.33). For this flow, the exact solutions are given by

$$\begin{aligned}
u &= \frac{1}{2\sqrt{x^2 + y^2}} \left[-3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right] \sqrt{x + \sqrt{x^2 + y^2}} \\
v &= \frac{1}{2\sqrt{x^2 + y^2}} \left[-3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right] \sqrt{-x + \sqrt{x^2 + y^2}} \\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\
p &= \frac{1}{4\sqrt{x^2 + y^2}} \left\{ -12\mu c_1 \sqrt{x + \sqrt{x^2 + y^2}} - \rho \left[-3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right]^2 \right\} \\
&\quad + p_0 \\
\omega &= -\frac{3c_1}{\sqrt{x^2 + y^2}} \sqrt{-x + \sqrt{x^2 + y^2}} \\
\Omega &= \beta(\psi)\omega - \frac{1}{2\sqrt{x^2 + y^2}} \left[-3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right]^2 \beta'(\psi)
\end{aligned} \tag{5.71}$$

where p_0 is an arbitrary constant and $\beta(\psi)$ is given by equation (5.69). If $c_1 = 0$, then the flow is irrotational. Thus, we have the following theorem:

Theorem 5.7. *Steady plane flow along $\eta = \text{constant}$ is permissible for infinitely conducting MHD aligned flow and the exact solutions for the rotational flow are given by equations (5.71) and for the irrotational flow by equations (5.71) with $c_1 = 0$.*

5.4.1.6 Example VI. (Flow with $\xi = \text{constant}$ as streamlines).

This example also deals with a streamline pattern for Hamel's problem and this pattern is not one of the four well known patterns. Figure 5.6 shows this flow pattern.

We let

$$\xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{5.72}$$

where $\gamma(\psi)$ is an unknown function of ψ and η is given by equation (5.33). Proceeding as above, we find that this is a permissible streamline pattern for infinitely

conducting MHD aligned flow and the solutions are given by

$$\begin{aligned}
\psi &= c_1 \xi^3 + c_3 \xi + c_4 = c_1 \left(\sqrt{x + \sqrt{x^2 + y^2}} \right)^3 + c_3 \sqrt{x + \sqrt{x^2 + y^2}} + c_4 \\
\beta^2(\psi) &= \frac{1}{\mu^*} \left[\rho - \frac{\beta_0}{\left(3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right)^2} \right] \\
u &= \frac{1}{2\sqrt{x^2 + y^2}} \left[3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right] \sqrt{-x + \sqrt{x^2 + y^2}} \\
v &= -\frac{1}{2\sqrt{x^2 + y^2}} \left[3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right] \sqrt{x + \sqrt{x^2 + y^2}} \\
H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\
p &= \frac{1}{2\sqrt{x^2 + y^2}} \left\{ 6\mu c_1 \sqrt{-x + \sqrt{x^2 + y^2}} - \frac{1}{2}\rho \left[3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right]^2 \right\} + p_0 \\
\omega &= -\frac{3c_1}{\sqrt{x^2 + y^2}} \sqrt{x + \sqrt{x^2 + y^2}} \\
\Omega &= \beta(\psi)\omega - \frac{1}{2\sqrt{x^2 + y^2}} \left[3c_1 x + 3c_1 \sqrt{x^2 + y^2} + c_3 \right]^2 \beta'(\psi)
\end{aligned} \tag{5.73}$$

where c_1, c_3, c_4, β_0 and p_0 are arbitrary constants. If $c_1 = 0$, then the flow is irrotational.

5.4.1.7 Example VII. (Flow with $\eta - \xi^3 = \text{constant}$ as streamlines).

We assume that

$$\eta - \xi^3 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{5.74}$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by equations (5.33). Following the same procedure as in previous examples, we conclude that this family of curves is a permissible streamline pattern for infinitely conducting MHD aligned flow and

the solutions are given by

$$\begin{aligned}
\gamma(\psi) &= d_1\psi + d_2 \\
\beta(\psi) &= \sqrt{\frac{\rho}{\mu^*}} \\
u &= \frac{1}{2d_1\sqrt{x^2+y^2}} (1-3y) \sqrt{x+\sqrt{x^2+y^2}} \\
v &= \frac{1}{2d_1\sqrt{x^2+y^2}} \left[\sqrt{-x+\sqrt{x^2+y^2}} + 3 \left(\sqrt{x+\sqrt{x^2+y^2}} \right)^3 \right] \\
H_1 &= \sqrt{\frac{\rho}{\mu^*}} u, \quad H_2 = \sqrt{\frac{\rho}{\mu^*}} v \\
p &= \frac{1}{2d_1\sqrt{x^2+y^2}} \left\{ -6\mu\sqrt{-x+\sqrt{x^2+y^2}} - \frac{\rho}{2d_1} \left[1 + 9 \left(x + \sqrt{x^2+y^2} \right)^2 \right] \right\} + p_0 \\
\omega &= \frac{3}{d_1\sqrt{x^2+y^2}} \sqrt{x+\sqrt{x^2+y^2}} \\
\Omega &= \sqrt{\frac{\rho}{\mu^*}} \omega
\end{aligned} \tag{5.75}$$

where p_0 is an arbitrary constant. The flow pattern for this example is shown in Figure 5.7.

5.4.2 Examples for $w = \ln z$.

Let $z = e^w$ or $w = \ln z$. Then, we have

$$\begin{aligned}
x &= e^\xi \cos \eta \\
y &= e^\xi \sin \eta
\end{aligned} \tag{5.76}$$

or

$$\begin{aligned}
\xi &= \ln r = \frac{1}{2} \ln (x^2 + y^2) \\
\eta &= \theta = \tan^{-1} \left(\frac{y}{x} \right)
\end{aligned} \tag{5.77}$$

Using equation (5.76) in (5.11), we obtain

$$J^* = e^{2\xi} \tag{5.78}$$

5.4.2.1 Example VIII. (Flow with $\eta - \xi - \xi^2 - e^{2\xi} = \text{constant}$ as streamlines).

We assume that

$$\eta - \xi - \xi^2 - e^{2\xi} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.79)$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by (5.77).

The streamlines are shown in Figure 5.8.

Comparing (5.79) with (5.16), we get

$$f(\xi) = \xi + \xi^2 + e^{2\xi}, \quad g(\xi) = 1 \quad (5.80)$$

Employing (5.22), (5.23), (5.78) and (5.80) in equations (5.20) and (5.21), we find that equation (5.21) is identically satisfied and equation (5.20) reduces to

$$\begin{aligned} \sum_{n=0}^4 A_n(\psi) \xi^n + \sum_{n=1}^4 A_{4+n}(\psi) e^{2n\xi} + \sum_{n=1}^3 A_{8+n}(\psi) \xi e^{2n\xi} \\ + \sum_{n=1}^2 A_{11+n}(\psi) \xi^2 e^{2n\xi} + A_{14}(\psi) \xi^3 e^{2\xi} = 0 \end{aligned} \quad (5.81)$$

where $A_n(\psi)$ are given by

$$\begin{aligned} A_0(\psi) &= 8 - 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ &\quad + \frac{4}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} \\ A_1(\psi) &= -32 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 16 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \end{aligned}$$

$$A_2(\psi) = 16 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 32 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{8}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$A_3(\psi) = 32 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 32 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_4(\psi) = 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_5(\psi) = -48 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - \frac{8}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$A_6(\psi) = 32 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 64 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 32 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - \frac{8}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi)\beta'(\psi)}{\gamma'^2(\psi)} \right\}$$

$$A_7(\psi) = -16 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 32 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_8(\psi) = 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_9(\psi) = -96 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 64 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_{10}(\psi) = -96 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 96 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_{11}(\psi) = 64 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_{12}(\psi) = 8 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 96 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_{13}(\psi) = 96 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

$$A_{14}(\psi) = 64 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]'$$

Since equation (5.81) must hold true for all values of ξ , then all the coefficients must vanish simultaneously. This requirement implies that

$$\gamma'(\psi) = -\frac{1}{2\mu} [\rho - \mu^* \beta_0^2] \psi + \psi_0$$

$$\beta(\psi) = \beta_0; \quad \beta_0 \neq \sqrt{\frac{\rho}{\mu^*}} \quad (5.82)$$

where ψ_0 and $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ are arbitrary constants. The solutions for this flow are given by

$$u = -\frac{2\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{x^2 + y^2} [x - y - y \ln(x^2 + y^2) - 2y(x^2 + y^2)]$$

$$v = \frac{2\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{x^2 + y^2} [-y - x - x \ln(x^2 + y^2) - 2x(x^2 + y^2)]$$

$$H_1 = \beta_0 u, \quad H_2 = \beta_0 v$$

$$p = -\frac{4\mu^2}{[\rho - \mu^* \beta_0^2]} [1 - 2(x^2 + y^2)] \frac{1}{x^2 + y^2} \ln(x^2 + y^2) - \frac{8\mu^2}{[\rho - \mu^* \beta_0^2]} \frac{1}{x^2 + y^2}$$

$$- \frac{16\mu^2}{[\rho - \mu^* \beta_0^2]} \left[\tan^{-1} \left(\frac{y}{x} \right) - \frac{1}{2} \ln(x^2 + y^2) - \frac{1}{4} \{ \ln(x^2 + y^2) \}^2 - (x^2 + y^2) \right]$$

$$- \frac{2\rho\mu^2}{[\rho - \mu^* \beta_0^2]^2} \frac{1}{x^2 + y^2} \left\{ 2 + [\ln(x^2 + y^2)]^2 + (2 + 4x^2 + 4y^2) \ln(x^2 + y^2) \right.$$

$$\left. + 4(x^2 + y^2)^2 \right\} + p_0$$

$$\omega = -\frac{2\mu}{[\rho - \mu^* \beta_0^2]} \left[4 + \frac{2}{x^2 + y^2} \right]$$

$$\Omega = \beta_0 \omega \quad (5.83)$$

where p_0 is an arbitrary constant. Even though mathematically this flow is possible, physically is not possible because the pressure is not single-valued function.

5.4.2.2 Example IX. (Flow along $\eta - \xi^2 = \text{constant}$ as streamlines).

We take

$$\eta - \xi^2 = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.84)$$

where $\gamma(\psi)$ is an unknown function and ξ, η are given by equations (5.77). Comparing (5.84) with equation (5.16), we have

$$f(\xi) = \xi^2, \quad g(\xi) = 1 \quad (5.85)$$

Employing (5.22), (5.23), (5.78) and (5.85) in equations (5.20) and (5.21), we find that the latter is identically satisfied and the former gives

$$\sum_{n=0}^4 B_n(\psi) \xi^n = 0 \quad (5.86)$$

where

$$\begin{aligned} B_0(\psi) &= 8 + 16 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 4 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ &\quad + \frac{4}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} + \frac{2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi) \beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ B_1(\psi) &= -48 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{8}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi) \beta'(\psi)}{\gamma'^2(\psi)} \right\} \\ B_2(\psi) &= 16 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 48 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 8 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ &\quad + \frac{8}{\mu} [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ B_3(\psi) &= 32 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \\ B_4(\psi) &= 16 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \end{aligned}$$

Requiring all the coefficients of equation (5.86) to vanish simultaneously, we obtain

$$\gamma(\psi) = -\frac{1}{2\mu} [\rho - \mu^* \beta_0^2] \psi + \psi_0, \quad \beta(\psi) = \beta_0 \quad (5.87)$$

where ψ_0 and $\beta_0 \neq \sqrt{\frac{\rho}{\mu^*}}$ are arbitrary constants. Hence, the given family of curves is a permissible streamline pattern for infinitely conducting MHD aligned flow and the exact solutions for this flow are given by

$$\begin{aligned}
u &= -\frac{2\mu}{[\rho - \mu^* \beta_0^2]} \left[\frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} \ln(x^2 + y^2) \right] \\
v &= -\frac{2\mu}{[\rho - \mu^* \beta_0^2]} \left[\frac{y}{x^2 + y^2} + \frac{x}{x^2 + y^2} \ln(x^2 + y^2) \right] \\
H_1 &= \beta_0 u, \quad H_2 = \beta_0 v \\
p &= -\frac{4\mu^2}{[\rho - \mu^* \beta_0^2]} \frac{1}{x^2 + y^2} [\ln(x^2 + y^2) + 1] \\
&\quad - \frac{2\rho\mu^2}{[\rho - \mu^* \beta_0^2]^2} \frac{1}{x^2 + y^2} \left\{ 1 + [\ln(x^2 + y^2)]^2 \right\} + p_0 \\
\omega &= -\frac{4\mu}{[\rho - \mu^* \beta_0^2]} \frac{1}{x^2 + y^2} \\
\Omega &= \beta_0 \omega
\end{aligned} \tag{5.88}$$

where p_0 is an arbitrary constant. The flow pattern for this example is shown in Figure 5.9.

5.4.2.3 Example X. (Flow with $\eta = \text{constant}$ as streamlines).

This example is a possible streamline pattern for the Hamel's problem for our fluid flow. This pattern is given in Figure 5.10.

We assume that

$$\eta = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \tag{5.89}$$

where $\gamma(\psi)$ is an arbitrary function of ψ and η is given by equation (5.77). Comparing (5.89) with equation (5.16), we get

$$f(\xi) = 0, \quad g(\xi) = 1 \tag{5.90}$$

Employing (5.22), (5.23), (5.78) and (5.90) in equations (5.20) and (5.21), we find that equation (5.21) is identically satisfied and equation (5.20) reduces to

$$4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right] + \frac{2}{\mu} \left\{ [\rho - \mu^* \beta^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \mu^* \frac{\beta(\psi) \beta'(\psi)}{\gamma'^2(\psi)} \right\} = 0 \tag{5.91}$$

Integrating (5.91) with respect to ψ , we obtain

$$\beta^2(\psi) = \frac{1}{\mu^*} \left[\rho + 4\mu\gamma'(\psi) - \mu\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + \mu\beta_0\gamma'^2(\psi) \right] \quad (5.92)$$

where β_0 is an arbitrary constant and $\gamma(\psi)$ is an arbitrary function of ψ . Thus, this family of streamlines is allowed by infinitely conducting MHD aligned flow and the exact solutions for this rotational flow are given by

$$\begin{aligned} u &= \frac{1}{\gamma'(\psi)} \frac{x}{x^2 + y^2} \\ v &= \frac{1}{\gamma'(\psi)} \frac{y}{x^2 + y^2} \\ H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\ p &= \frac{1}{2(x^2 + y^2)} \left[\frac{\mu}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' - \frac{\rho}{\gamma'^2(\psi)} \right] + p_0 \\ \omega &= \frac{1}{x^2 + y^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ \Omega &= \beta(\psi)\omega - \frac{1}{x^2 + y^2} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \end{aligned} \quad (5.93)$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is given by equation (5.92) and $\gamma(\psi)$ is an arbitrary function of ψ .

5.4.2.4 Example XI. (Flow with $\xi = \text{constant}$ as streamlines).

The flow pattern in this example is a possible solution of the Hamel's problem for our flow. Figure 5.11 is shown this streamline pattern.

We let

$$\xi = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (5.94)$$

where $\gamma(\psi)$ is an unknown function and ξ is given by (5.77). Using (5.94) in (5.78), we get

$$J^* = e^{2\gamma(\psi)} \quad (5.95)$$

Comparing (5.94) with (5.24), we obtain

$$k(\eta) = 0, \quad m(\eta) = 1 \quad (5.96)$$

Employing (5.30), (5.31), (5.95) and (5.96) in equations (5.28) and (5.29), we find that equation (5.29) is identically satisfied and equation (5.28) gives

$$\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - 4 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 4 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0 \quad (5.97)$$

Thus, $\xi = \text{constant}$ can serve as streamline pattern for infinitely conducting MHD aligned flow and the solutions are given by

$$\begin{aligned} u &= \frac{1}{\gamma'(\psi)} \frac{y}{x^2 + y^2} \\ v &= -\frac{1}{\gamma'(\psi)} \frac{x}{x^2 + y^2} \\ H_1 &= \beta(\psi)u, \quad H_2 = \beta(\psi)v \\ p &= \frac{\mu}{\gamma'(\psi)} \left[e^{-2\gamma(\psi)} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' \theta + \frac{1}{2} e^{-2\gamma(\psi)} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'^2(\psi)} \\ &\quad + \int e^{-2\gamma(\psi)} [\rho - \mu^* \beta^2(\psi)] \frac{1}{\gamma'(\psi)} d\psi - \frac{\rho}{2\gamma'^2(\psi)} \frac{1}{x^2 + y^2} + p_0 \\ \omega &= \frac{1}{x^2 + y^2} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\ \Omega &= \beta(\psi)\omega - \frac{1}{x^2 + y^2} \frac{\beta'(\psi)}{\gamma'^2(\psi)} \end{aligned} \quad (5.98)$$

where p_0 is an arbitrary constant, $\beta(\psi)$ is an arbitrary function of ψ and $\gamma(\psi)$ is a solution of equation (5.97). Requiring the pressure to be single-valued, we must take

$$\left[e^{-2\gamma(\psi)} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right]' = 0$$

which, upon integration, gives

$$a_1 e^{2\gamma(\psi)} + 2a_2 \gamma(\psi) + a_3 - \psi = 0 \quad (5.99)$$

where a_1 , a_2 and a_3 are arbitrary constants that are not simultaneously zero. Using (5.99), equation (5.97) is identically satisfied. Employing (5.94) in (5.99), we obtain

$$\psi = a_1 e^{2\xi} + 2a_2 \xi + a_3 \quad (5.100)$$

Using equation (5.77) in (5.100), we obtain

$$\psi = a_1(x^2 + y^2) + a_2 \ln(x^2 + y^2) + a_3$$

Hence, the solutions for this rotational flow are given by equations (5.98) with $\gamma(\psi)$ given implicitly by equation (5.99). If $a_1 = 0$, then the flow is irrotational.

Letting $\beta^2(\psi) = \frac{1}{\mu^*} [\rho - \gamma'(\psi)]$ and using (5.99), the solutions (5.98) take the form

$$u = 2a_1 y + \frac{2a_2 y}{x^2 + y^2}$$

$$v = - \left(2a_1 x + \frac{2a_2 x}{x^2 + y^2} \right)$$

$$H_1 = \beta u, \quad H_2 = \beta v$$

$$p = - \frac{\rho}{2(x^2 + y^2)} [2a_1(x^2 + y^2) + 2a_2]^2 + p_0$$

$$\omega = -4a_1$$

$$\Omega = \beta \omega - \frac{\beta'(\psi)}{x^2 + y^2} [2a_1(x^2 + y^2) + 2a_2]^2$$

where $\beta^2(\psi) = \frac{1}{\mu^*} [\rho - \gamma'(\psi)]$.

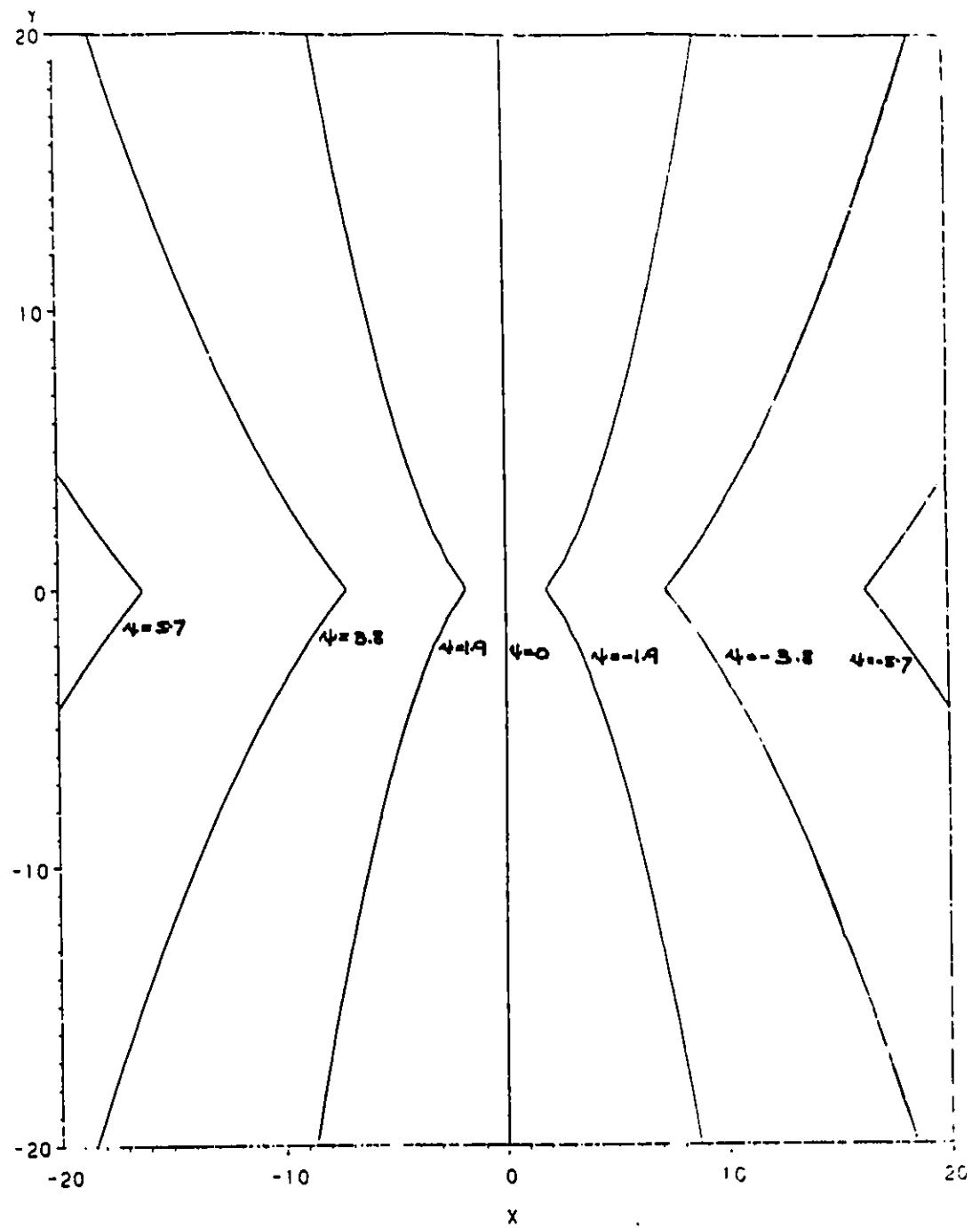


FIGURE 5.1: Streamline pattern for $\eta(z, y) - \xi(z, y) = \text{constant}$
when $\sqrt{2z} = \xi(z, y) + i\eta(z, y)$.

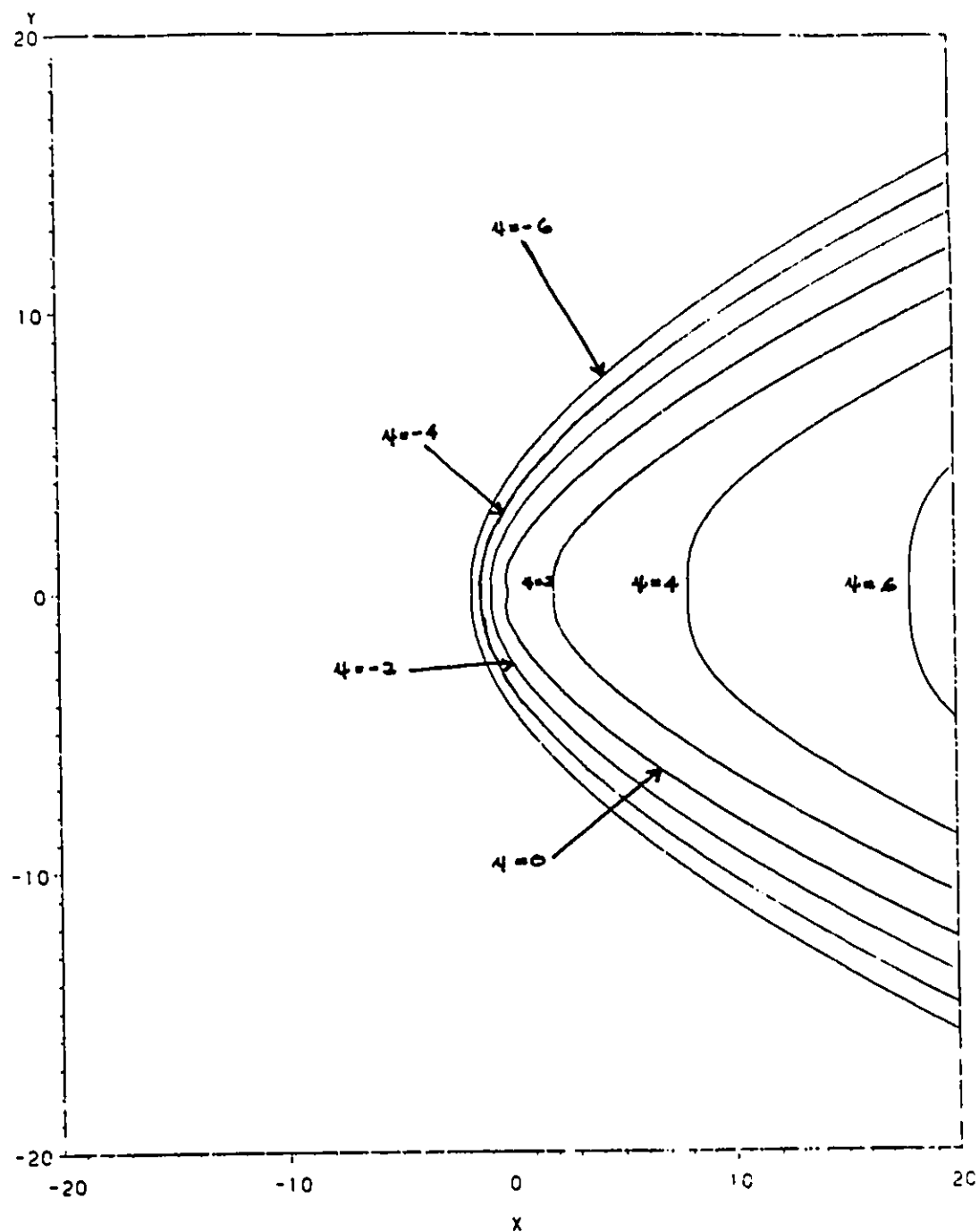


FIGURE 5.2: Streamline pattern for $\xi(z, y) - \eta^3(z, y) = \text{constant}$
when $\sqrt{2}z = \xi(z, y) + i\eta(z, y)$.

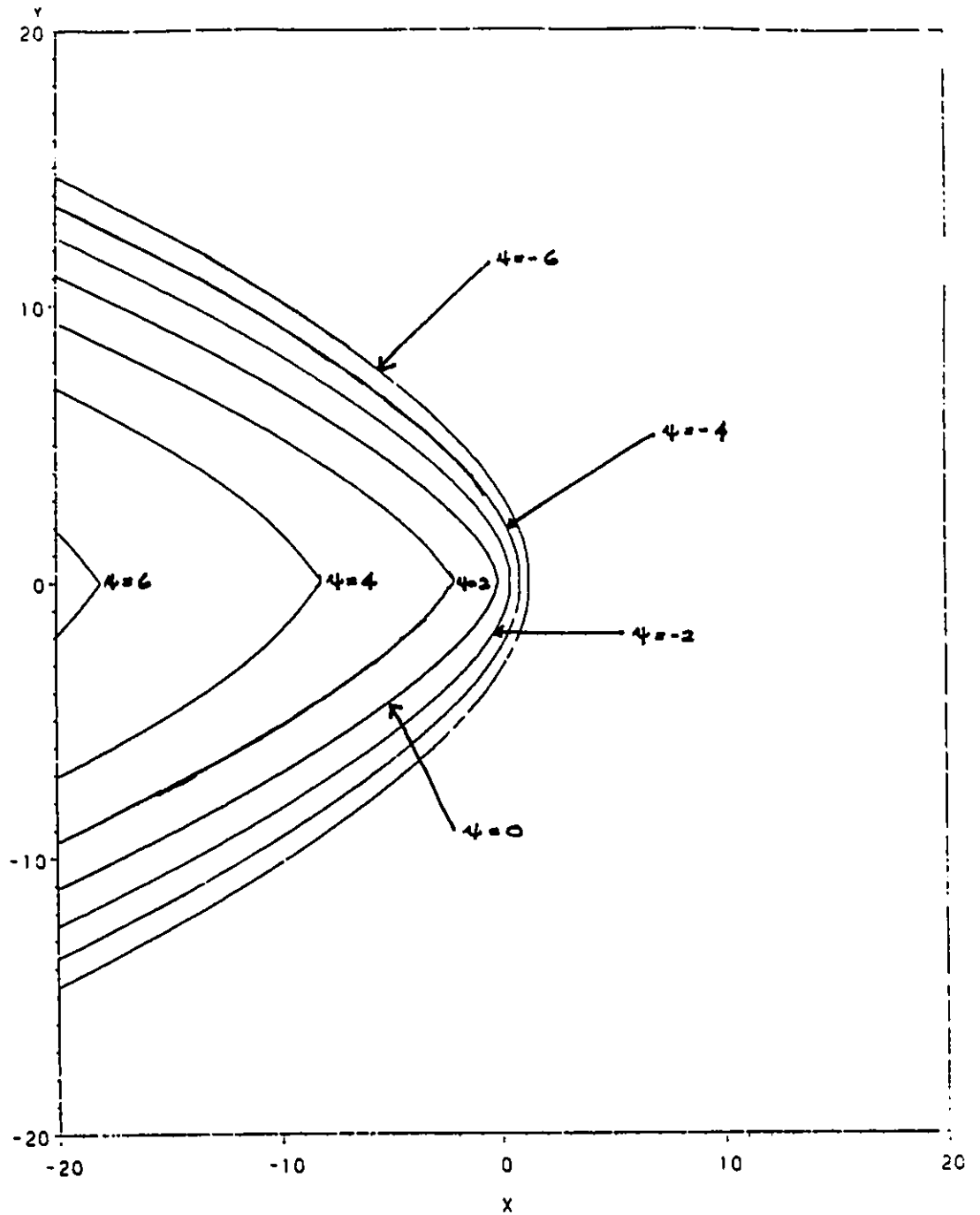


FIGURE 5.3: Streamline pattern for $\eta(x, y) - \xi(x, y) - \xi^3(x, y) = \text{constant}$ when $\sqrt{2}z = \xi(x, y) + i\eta(x, y)$.

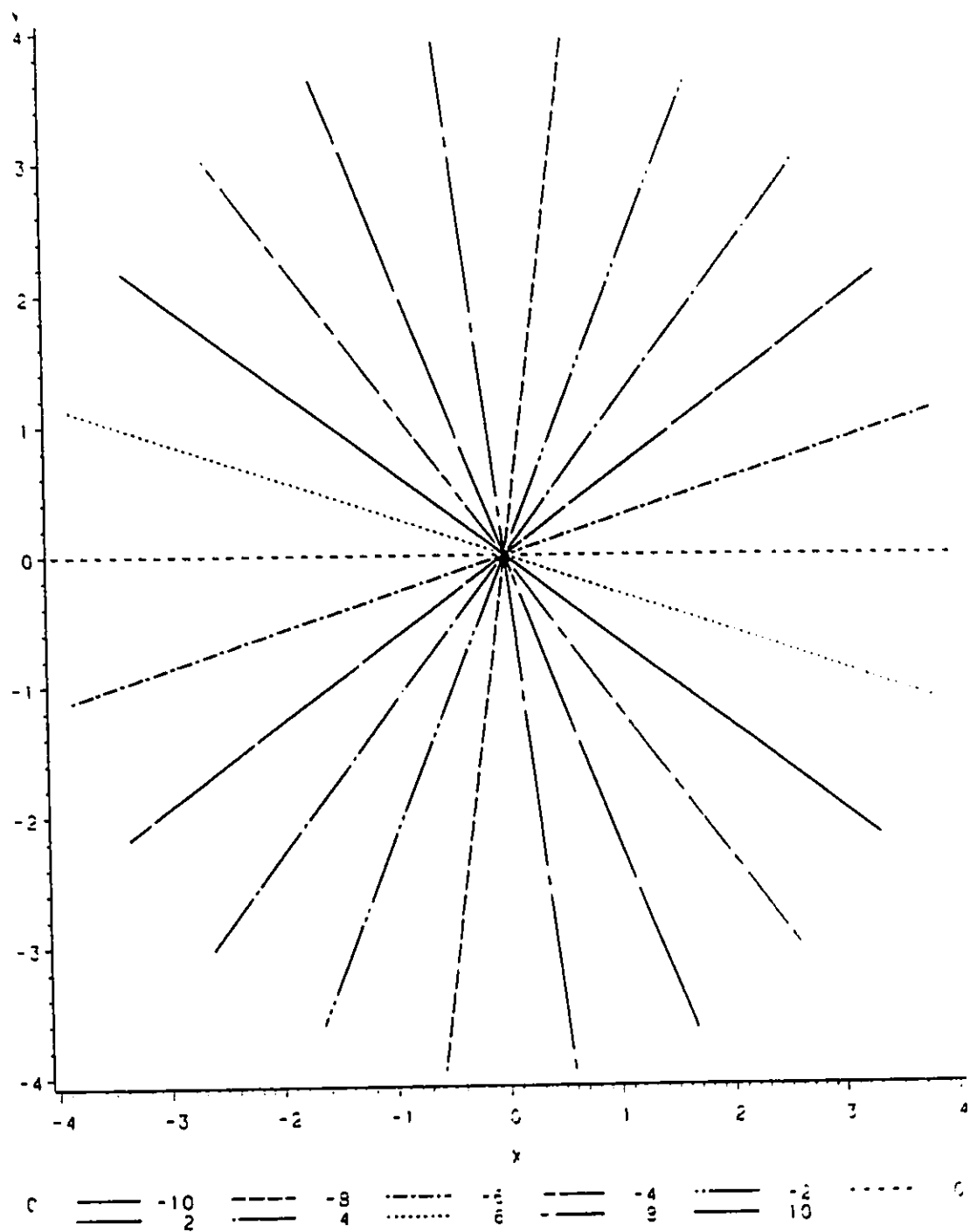


FIGURE 5.4: Streamline pattern for $\frac{\eta(z,y)}{\xi(z,y)} = \text{constant}$ when $\sqrt{2}z = \xi(z,y) + i\eta(z,y)$.

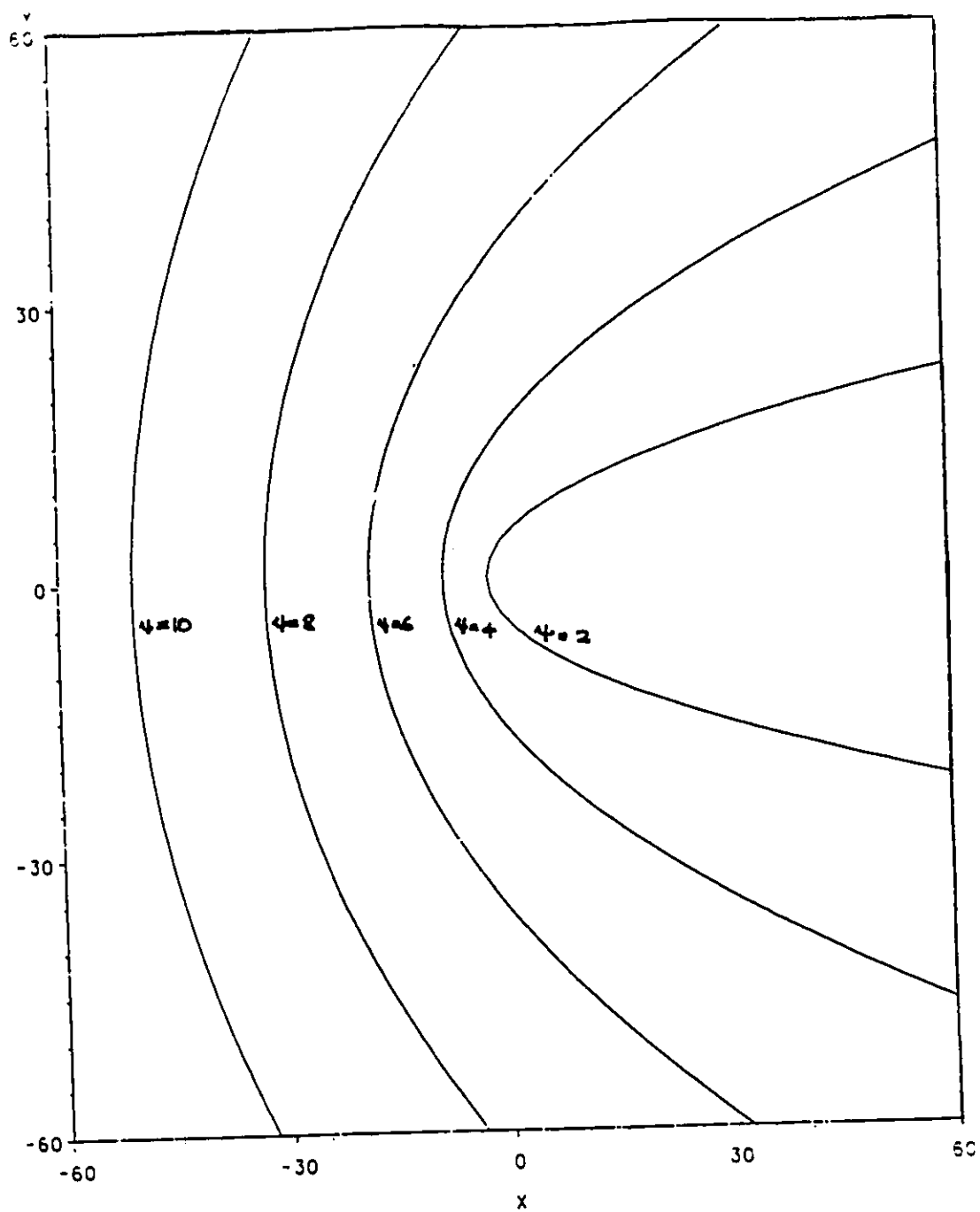


FIGURE 5.5: Streamline pattern for $\eta(x, y) = \text{constant}$
when $\sqrt{2}z = \xi(x, y) + i\eta(x, y)$.

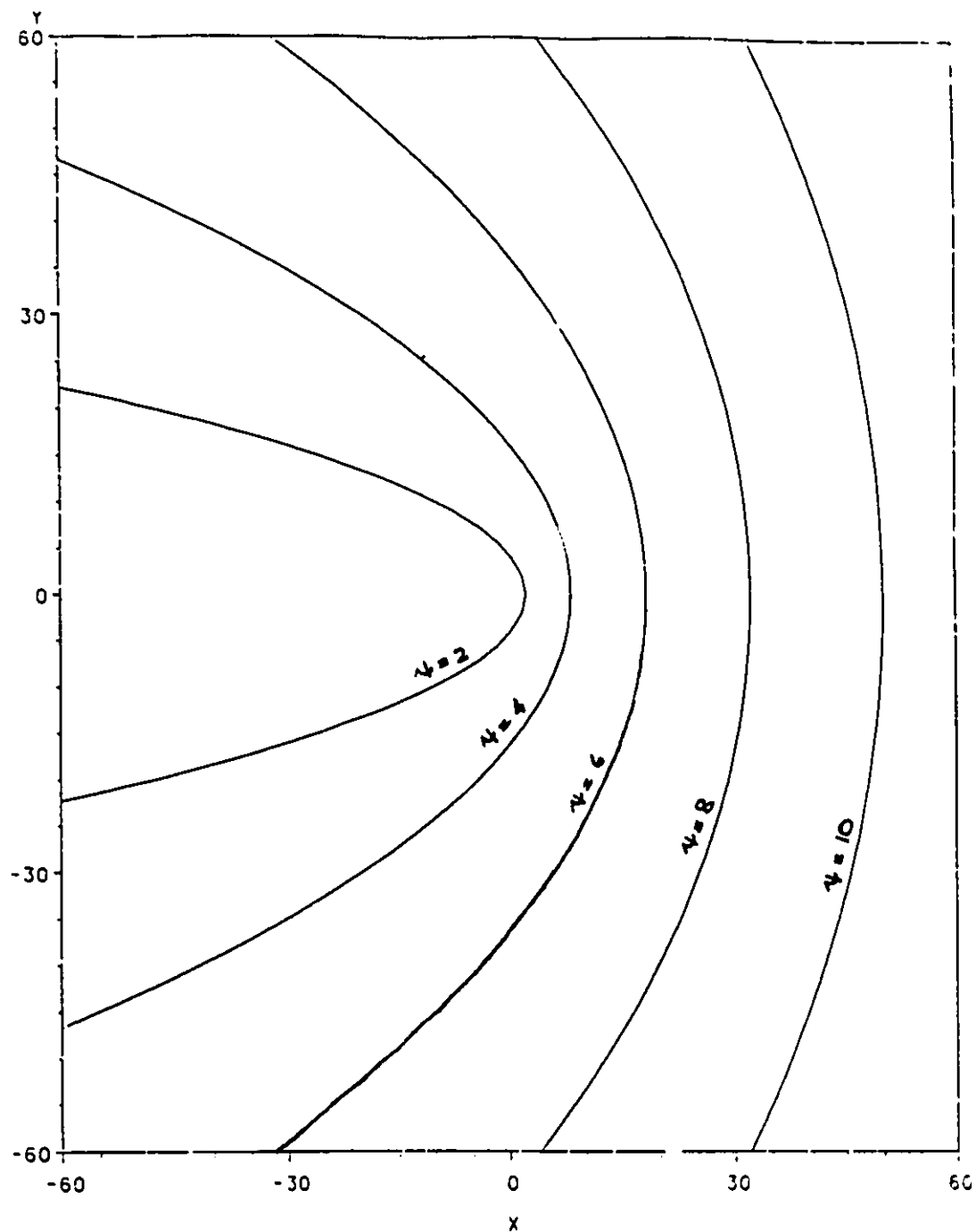


FIGURE 5.6: Streamline pattern for $\xi(x, y) = \text{constant}$
when $\sqrt{2z} = \xi(x, y) + i\eta(x, y)$.

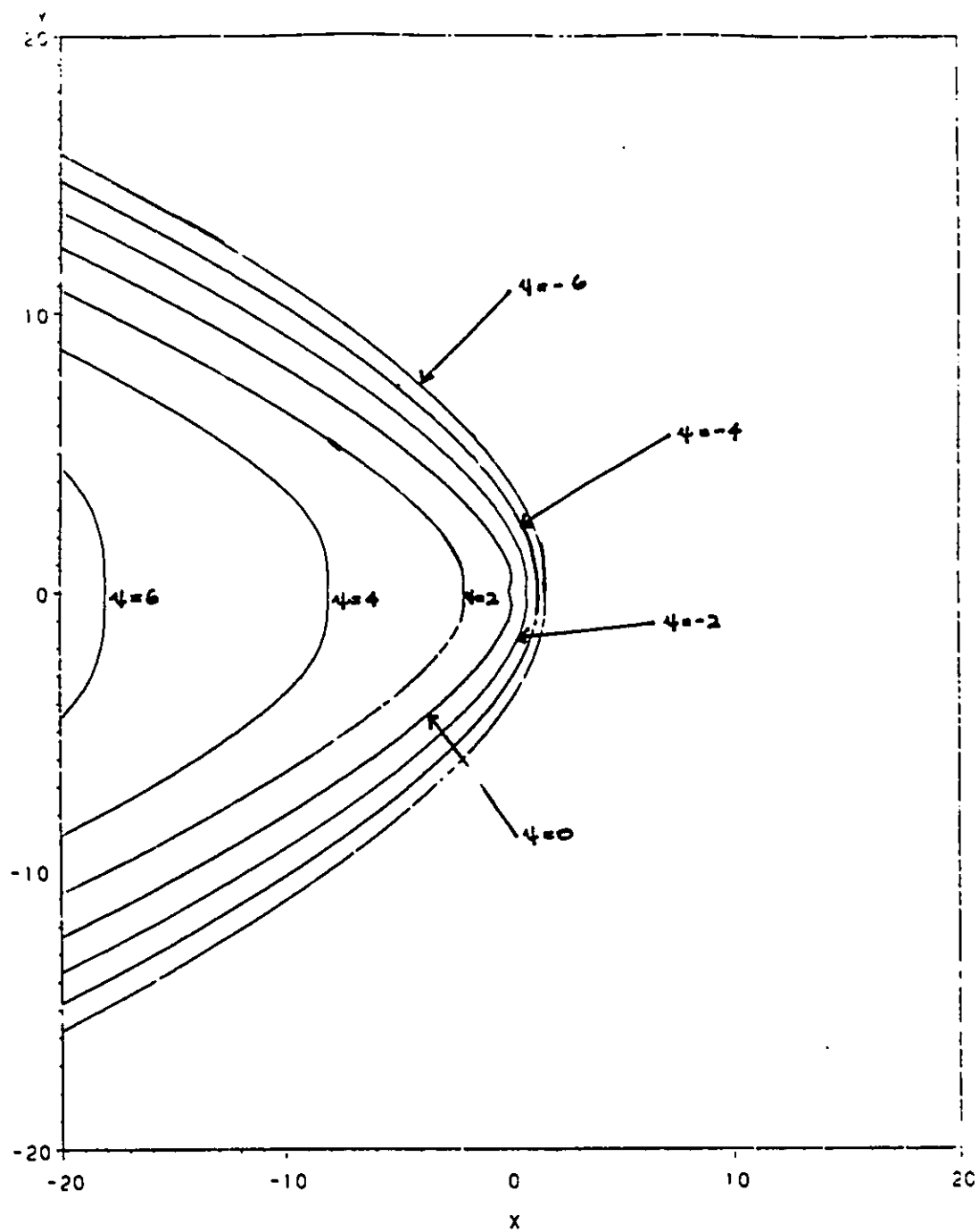


FIGURE 5.7: Streamline pattern for $\eta(x, y) - \xi^3(x, y) = \text{constant}$ when $\sqrt{2}z = \xi(x, y) + i\eta(x, y)$.

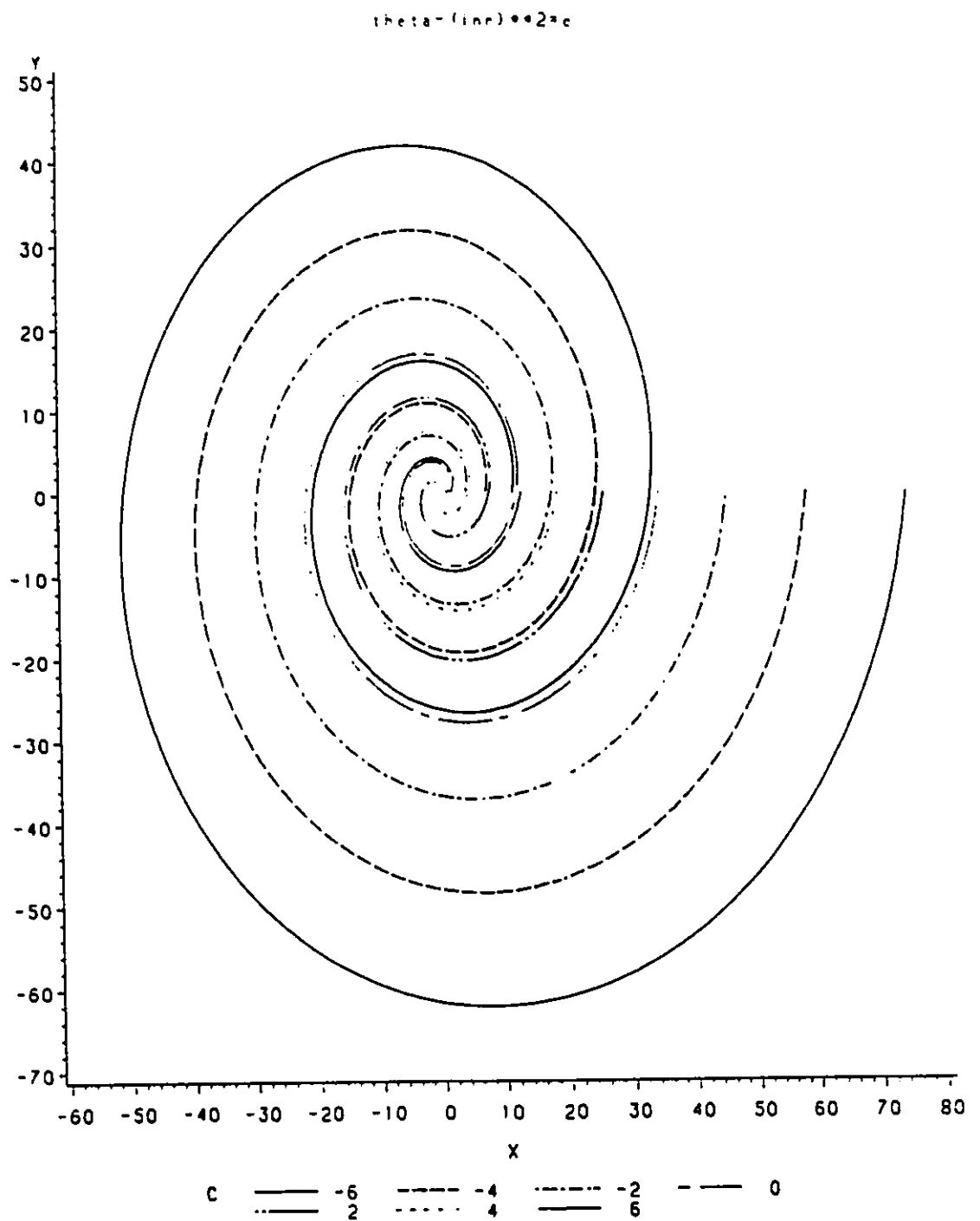


FIGURE 5.9: Streamline pattern for $\eta(x, y) - \xi^2(x, y) = \text{constant}$ when $\ln z = \xi(x, y) + i\eta(x, y)$.

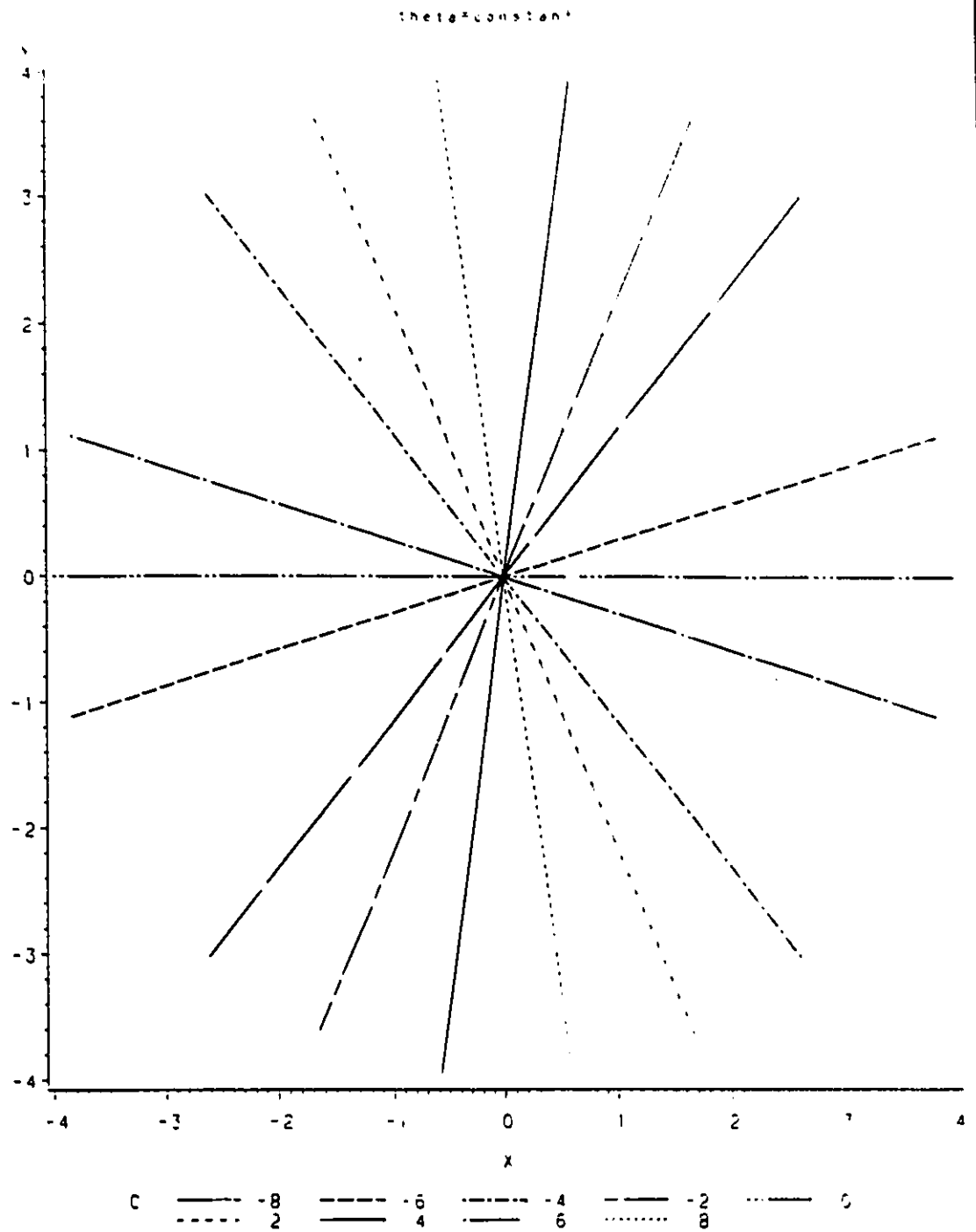


FIGURE 5.10: Streamline pattern for $\eta(x, y) = \text{constant}$
when $\ln z = \xi(x, y) + i\eta(x, y)$.

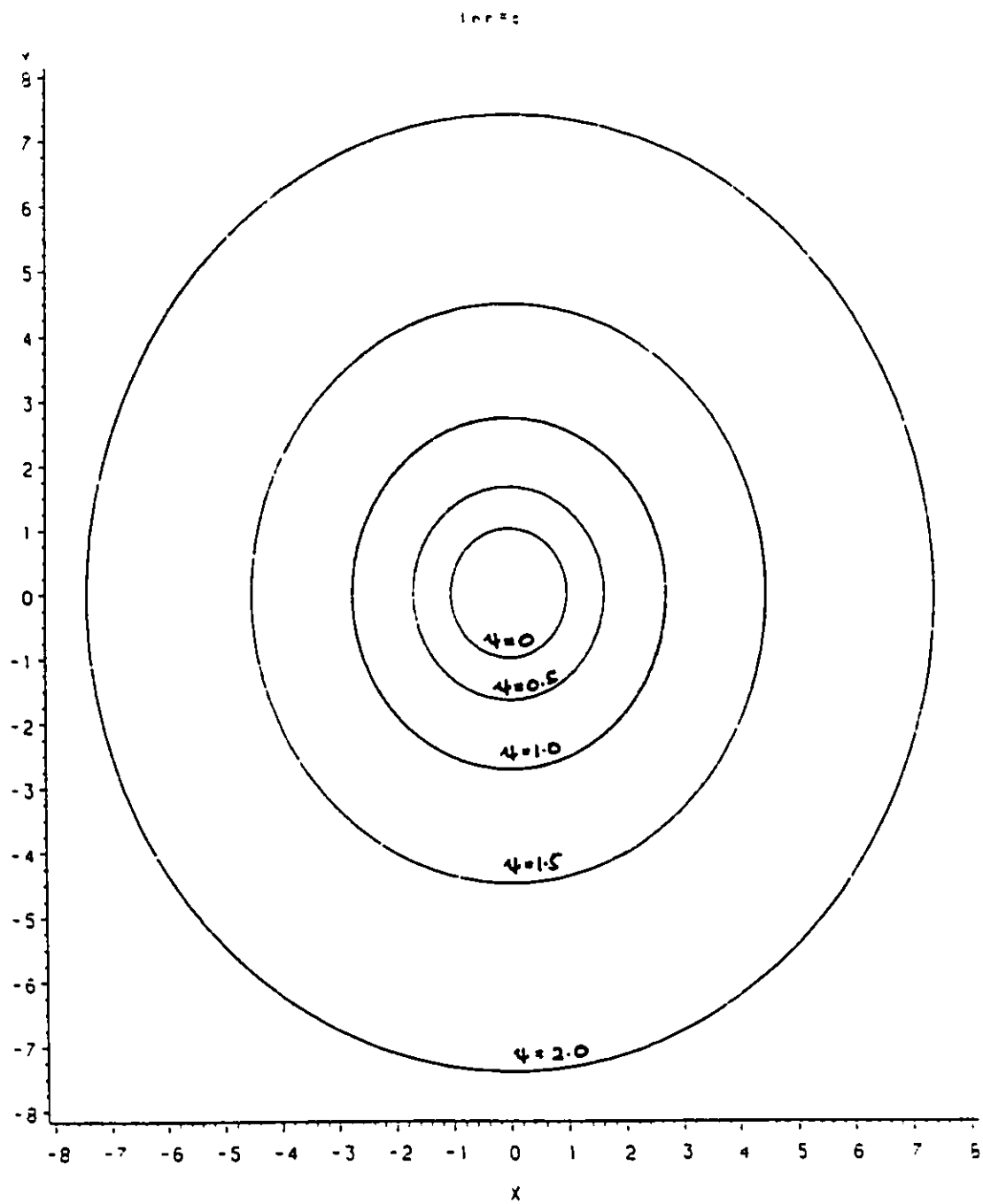


FIGURE 5.11: Streamline pattern for $\xi(z,y) = \text{constant}$ when $\ln z = \xi(z,y) + i\eta(z,y)$.

CHAPTER 6

SOLUTIONS OF ORTHOGONAL AND VARIABLY-INCLINED MHD FLOWS USING VON MISES COORDINATES

6.1 INTRODUCTION.

This chapter deals with steady plane infinitely conducting MHD orthogonal and variably-inclined flows. A MHD flow is said to be *orthogonal* or *crossed* if the magnetic field vector is perpendicular to the velocity field vector everywhere in the flow plane. A MHD flow is said to be *variably-inclined* if the angle between the magnetic field vector and the velocity field vector varies from point to point in the flow plane.

For steady plane infinitely conducting MHD orthogonal flow, we find all possible functions $f(x)$ and $g(x)$ such that the family of curves $\frac{y - f(x)}{g(x)} = \text{constant}$ forms a streamline pattern and determine that the only two possible flows are the irrotational parallel and the radial flows.

For steady plane variably-inclined flow, we study the following flows:

(i) $y - m_1 x^2 - m_2 x = \text{constant},$

(ii) $\frac{y}{x} = \text{constant}.$

This study is carried out following the approach developed in Chapter III.

6.2 FLOW EQUATIONS.

The steady plane flow of a viscous incompressible and electrically conducting fluid, in the presence of a magnetic field, is governed by equations (2.10) to (2.18).

Taking $H_3 = 0$, assuming that the fluid flow is infinitely conducting ($\sigma \rightarrow \infty$) and introducing

$$h = \frac{1}{2}\rho(u^2 + v^2) + p \quad (6.1)$$

in equations (2.10) to (2.18), we obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6.2)$$

$$\frac{\partial h}{\partial x} + \mu \frac{\partial \omega}{\partial y} - \rho v \omega + \mu^* H_2 \Omega = 0 \quad (6.3)$$

$$\frac{\partial h}{\partial y} - \mu \frac{\partial \omega}{\partial x} + \rho u \omega - \mu^* H_1 \Omega = 0 \quad (6.4)$$

$$u H_2 - v H_1 = C \quad (6.5)$$

$$\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y} = 0 \quad (6.6)$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (6.7)$$

$$\Omega = \frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial y} \quad (6.8)$$

This is a system of seven equations in seven unknown functions $u(x, y)$, $v(x, y)$, $H_1(x, y)$, $H_2(x, y)$, $h(x, y)$, $\omega(x, y)$ and $\Omega(x, y)$.

We now consider variably-inclined plane flows and let $\theta(x, y)$ be the variable angle between the velocity field vector and the magnetic field vector such that $\theta(x, y) \neq 0$.

6.3 FLOW EQUATIONS IN MARTIN'S FORM.

These equations were previously derived by Chandna, Barron and Chew [1983].

Continuity equation (6.2) implies the existence of a streamfunction $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \quad (6.9)$$

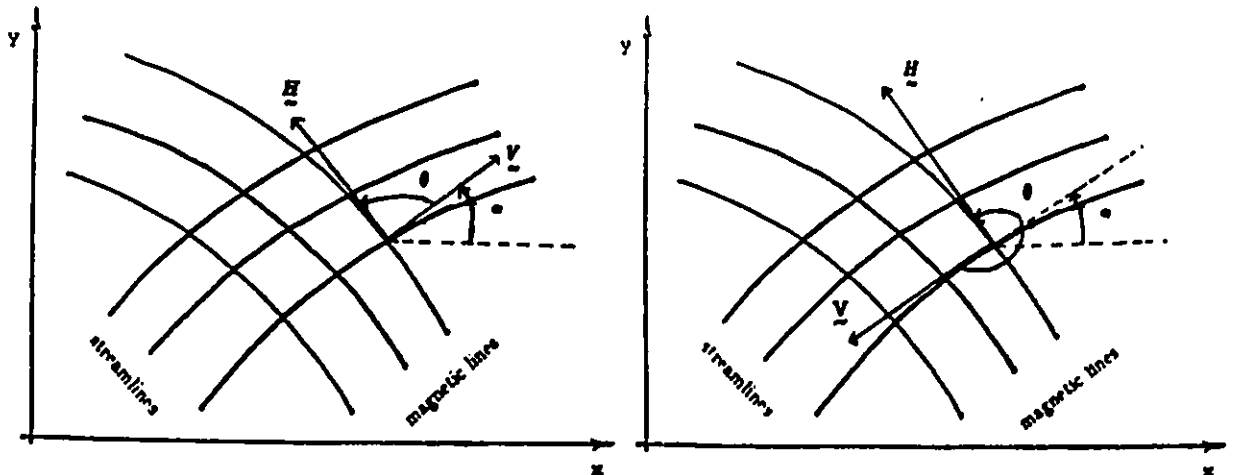
We take $\phi(x, y) = \text{constant}$ to be some arbitrary family of curves which generates with the streamlines $\psi(x, y) = \text{constant}$ a curvilinear net so that in the physical plane the independent variables x, y can be replaced by ϕ, ψ .

Having recorded the results from differential geometry in section 2.2, we follow and employ Martin's [1971] work and transform equations (6.2) to (6.8) into new form with new variables ϕ, ψ .

Let α be the angle of inclination of the tangent to the coordinate line $\psi = \text{constant}$ directed in the sense of increasing ϕ . The magnetic field vector H makes an angle $\theta + \alpha$ or $\theta + \alpha - \pi$ with the x -axis accordingly as fluid flows along the streamlines towards higher or lower parameter values of ϕ . Hence, we have

$$H_1 = \pm H \cos(\theta + \alpha), \quad H_2 = \pm H \sin(\theta + \alpha) \quad (6.10)$$

where the positive or negative sign is taken as J is positive or negative and $H = \sqrt{H_1^2 + H_2^2}$.



Linear Momentum Equations (6.3) and (6.4): Employing (6.9), (6.10) in the linear momentum equations (6.3) and (6.4) and making use of (2.26), we obtain

$$\frac{\partial h}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial y}{\partial \phi} + \mu \left(-\frac{\partial \omega}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial \omega}{\partial \psi} \frac{\partial x}{\partial \phi} \right) - \rho \omega \frac{\partial y}{\partial \phi} + \mu^* W \Omega H \sin(\theta + \alpha) = 0 \quad (6.11)$$

$$\frac{\partial h}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial x}{\partial \phi} + \mu \left(\frac{\partial \omega}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi} \frac{\partial y}{\partial \phi} \right) - \rho \omega \frac{\partial x}{\partial \phi} + \mu^* W \Omega H \cos(\theta + \alpha) = 0 \quad (6.12)$$

Multiplying (6.11) by $\frac{\partial x}{\partial \phi}$, (6.12) by $\frac{\partial y}{\partial \phi}$ and subtracting the resulting equations, we get

$$J \frac{\partial h}{\partial \phi} = \mu \left[F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right] - \mu^* W \Omega H \sqrt{E} \sin \theta$$

Multiplying (6.11) by $\frac{\partial x}{\partial \psi}$, (6.12) by $\frac{\partial y}{\partial \psi}$ and subtracting the resulting equations, we have

$$J \frac{\partial h}{\partial \psi} = \mu \left[-F \frac{\partial \omega}{\partial \psi} + G \frac{\partial \omega}{\partial \phi} \right] - \rho \omega J - \mu^* W \Omega H \left[\frac{F}{\sqrt{E}} \sin \theta - \frac{J}{\sqrt{E}} \cos \theta \right]$$

Solenoidal Equation (6.6): Using (2.26) in the solenoidal equation (6.6), it follows that

$$\left(\frac{\partial H_1}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial H_1}{\partial \psi} \frac{\partial y}{\partial \phi} \right) - \left(\frac{\partial H_2}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial H_2}{\partial \psi} \frac{\partial x}{\partial \phi} \right) = 0$$

Employing (2.28), (2.31) and (6.10) in this equation, we get

$$\begin{aligned} & \frac{\partial H}{\partial \phi} (-F \sin \theta + J \cos \theta) + \frac{\partial H}{\partial \psi} E \sin \theta - H (F \cos \theta + J \sin \theta) \frac{\partial \theta}{\partial \phi} \\ & + H E \cos \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \cos \theta + J \sin \theta) + H J \Gamma_{12}^2 \cos \theta = 0 \end{aligned}$$

Current Density Equation (6.8): From (6.8) and (2.26), we have

$$J \Omega = \left(\frac{\partial H_2}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial H_2}{\partial \psi} \frac{\partial y}{\partial \phi} \right) + \left(\frac{\partial H_1}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial H_1}{\partial \psi} \frac{\partial x}{\partial \phi} \right)$$

which upon substitution of (2.28), (2.31) and (6.10) yields

$$\begin{aligned} \sqrt{E} W \Omega &= \frac{\partial H}{\partial \phi} (F \cos \theta + J \sin \theta) - \frac{\partial H}{\partial \psi} E \cos \theta - H (F \sin \theta - J \cos \theta) \frac{\partial \theta}{\partial \phi} \\ &+ H E \sin \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \sin \theta - J \cos \theta) + H J \Gamma_{12}^2 \sin \theta \end{aligned}$$

Equation of Continuity (6.2) and Vorticity Equation (6.7): As we mentioned previously, the continuity equation takes the form:

$$Wq = \sqrt{E}, \quad u + iv = \frac{\sqrt{E}}{W} e^{ia} \quad (6.13)$$

where $q = \sqrt{u^2 + v^2}$ is the speed.

The vorticity equation in (ϕ, ψ) -net becomes

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right]$$

Diffusion Equation (6.5): Since θ is the angle of the magnetic field vector and the velocity vector, then equation (6.5) can be written as

$$qH \sin \theta = C$$

Using (6.13), this equation reduces to

$$\sqrt{E}H \sin \theta = CW$$

Summing up the above results, we have:

Theorem 6.1. *If the streamlines $\psi(x, y) = \text{constant}$ and arbitrary family of curves $\phi(x, y) = \text{constant}$ generate a curvilinear net in the physical plane of a steady plane viscous incompressible and infinitely conducting MHD variably-inclined fluid flow,*

then the flow in independent variables ϕ, ψ is governed by the following system:

$$\begin{aligned}
J \frac{\partial h}{\partial \phi} &= \mu \left[F \frac{\partial \omega}{\partial \phi} - E \frac{\partial \omega}{\partial \psi} \right] - \mu^* W \Omega H \sqrt{E} \sin \theta \\
J \frac{\partial h}{\partial \psi} &= \mu \left[-F \frac{\partial \omega}{\partial \psi} + G \frac{\partial \omega}{\partial \phi} \right] - \rho \omega J - \mu^* W \Omega H \left[\frac{F}{\sqrt{E}} \sin \theta - \frac{J}{\sqrt{E}} \cos \theta \right] \\
\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) &= 0 \\
\omega &= \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \\
\frac{\partial H}{\partial \phi} (-F \sin \theta + J \cos \theta) + \frac{\partial H}{\partial \psi} E \sin \theta - H (F \cos \theta + J \sin \theta) \frac{\partial \theta}{\partial \phi} & \\
+ H E \cos \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \cos \theta + J \sin \theta) + H J \Gamma_{12}^2 \cos \theta &= 0 \\
\sqrt{E} W \Omega &= \frac{\partial H}{\partial \phi} (F \cos \theta + J \sin \theta) - \frac{\partial H}{\partial \psi} E \cos \theta - H (F \sin \theta - J \cos \theta) \frac{\partial \theta}{\partial \phi} \\
+ H E \sin \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \sin \theta - J \cos \theta) + H J \Gamma_{12}^2 \sin \theta & \\
\sqrt{E} H \sin \theta &= C W
\end{aligned} \tag{6.14}$$

of seven equations in eight unknowns $E, F, G, h, \omega, \Omega, H$ and θ as functions of ϕ, ψ . Given a solution of system (6.14), the pressure function is determined from

$$p = h - \frac{\rho}{2} \frac{E}{W^2}$$

We use the integrability condition $\frac{\partial^2 h}{\partial \phi \partial \psi} = \frac{\partial^2 h}{\partial \psi \partial \phi}$ to eliminate h from the linear momentum equations in system (6.14) and we note that the seven functions $E(\phi, \psi), F(\phi, \psi), G(\phi, \psi), \omega(\phi, \psi), \Omega(\phi, \psi), H(\phi, \psi)$ and $\theta(\phi, \psi)$ satisfy the following system:

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \tag{6.15}$$

$$\frac{\partial}{\partial \psi} \left(\frac{W}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{W}{E} \Gamma_{12}^2 \right) = 0 \tag{6.16}$$

$$\mu \left[\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial \omega}{\partial \phi} - \frac{F}{J} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} - \frac{F}{J} \frac{\partial \omega}{\partial \phi} \right) \right] - \rho \frac{\partial \omega}{\partial \phi} + \mu^* \left\{ \frac{\partial}{\partial \psi} \left[W \Omega H \frac{\sqrt{E}}{J} \sin \theta \right] + \frac{\partial}{\partial \phi} \left[W \Omega H \left(\frac{1}{\sqrt{E}} \cos \theta - \frac{F}{J \sqrt{E}} \sin \theta \right) \right] \right\} = 0 \quad (6.17)$$

$$\frac{\partial H}{\partial \phi} (-F \sin \theta + J \cos \theta) + \frac{\partial H}{\partial \psi} E \sin \theta - H (F \cos \theta + J \sin \theta) \frac{\partial \theta}{\partial \phi} + H E \cos \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \cos \theta + J \sin \theta) + H J \Gamma_{12}^2 \cos \theta = 0 \quad (6.18)$$

$$\sqrt{E} W \Omega = \frac{\partial H}{\partial \phi} (F \cos \theta + J \sin \theta) - \frac{\partial H}{\partial \psi} E \cos \theta - H (F \sin \theta - J \cos \theta) \frac{\partial \theta}{\partial \phi} + H E \sin \theta \frac{\partial \theta}{\partial \psi} - H \frac{J}{E} \Gamma_{11}^2 (F \sin \theta - J \cos \theta) + H J \Gamma_{12}^2 \sin \theta \quad (6.19)$$

$$\sqrt{E} H \sin \theta = C W \quad (6.20)$$

of six equations.

Equations (6.15) to (6.20) form an underdetermined system of six equations in seven unknowns. The reason for this is that the family of curves $\phi(x, y) = \text{constant}$ are left arbitrary. There are a number of different ways that we can make a determinate system. In the following work, we let $\phi(x, y) = \text{constant}$ be $x = \text{constant}$ so that we deal with the von Mises (x, ψ) -net.

If $\theta = \frac{\pi}{2}$ is taken in system of equations (6.15) to (6.20), then we have the flow equations of orthogonal MHD fluid flow and these equations were previously derived by Garg and Chandna [1976]. In their work, they took the ϕ -curves to coincide with the magnetic lines.

In the following sections, we assume that $J = W > 0$.

6.4. SOLUTIONS FOR MHD ORTHOGONAL FLOW.

Letting $\theta = \frac{\pi}{2}$ and $\phi = x$ in equations (6.15) to (6.20), we find that Gauss equation (6.15) is identically satisfied and the remaining equations reduce to

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial x} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] \quad (6.21)$$

$$\begin{aligned} \mu \left[\frac{\partial}{\partial x} \left(\frac{G}{J} \frac{\partial \omega}{\partial x} - \frac{F}{J} \frac{\partial \omega}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{\partial \omega}{\partial \psi} - \frac{F}{J} \frac{\partial \omega}{\partial x} \right) \right] - \rho \frac{\partial \omega}{\partial x} \\ + \mu^* \left\{ \frac{\partial}{\partial \psi} \left[\Omega H \sqrt{E} \right] - \frac{\partial}{\partial x} \left[\Omega H \frac{F}{\sqrt{E}} \right] \right\} = 0 \end{aligned} \quad (6.22)$$

$$-F \frac{\partial H}{\partial x} + E \frac{\partial H}{\partial \psi} - \frac{H J^2}{E} \Gamma_{11}^2 = 0 \quad (6.23)$$

$$\sqrt{E} J \Omega = J \frac{\partial H}{\partial x} - \frac{H J F}{E} \Gamma_{11}^2 + H J \Gamma_{12}^2 \quad (6.24)$$

$$\sqrt{E} H = C J \quad (6.25)$$

Equations (6.21) to (6.25) form a system of five equations in five unknowns, the unknowns being E, F, G, ω, H and Ω .

We want to find all functions $f(x)$ and $g(x)$, so that

$$\frac{y - f(x)}{g(x)} = \text{constant}$$

is a streamline pattern for infinitely conducting MHD orthogonal flow. If $\frac{y - f(x)}{g(x)} = \text{constant}$ are the streamlines, then there exists some function $\gamma(\psi)$ such that

$$\frac{y - f(x)}{g(x)} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (6.26)$$

where $\gamma'(\psi)$ denotes the derivative of $\gamma(\psi)$ with respect to ψ .

Substituting equation (6.26) and $\phi = x$ in (2.24) and (2.26), we have

$$\begin{aligned} E &= 1 + [f'(x) + g'(x)\gamma(\psi)]^2, \quad G = g^2(x)\gamma'^2(\psi) \\ F &= [f'(x) + g'(x)\gamma(\psi)]g(x)\gamma'(\psi), \quad J = W = g(x)\gamma'(\psi) \end{aligned} \quad (6.27)$$

Using (6.27) and $\phi = x$ in equations (2.34), we get

$$\begin{aligned}\Gamma_{11}^2 &= \frac{1}{g(x)\gamma'(\psi)} [f''(x) + g''(x)\gamma(\psi)] \\ \Gamma_{12}^2 &= \frac{g'(x)}{g(x)}\end{aligned}\tag{6.28}$$

Employing (6.27), equation (6.25) yields

$$H = \frac{Cg(x)\gamma'(\psi)}{\sqrt{1 + [f'(x) + g'(x)\gamma(\psi)]^2}}\tag{6.29}$$

Substituting (6.27) to (6.29) in equation (6.23), we obtain

$$\begin{aligned}&\left\{ [f'(x) + g'(x)\gamma(\psi)]^2 [f''(x) + g''(x)\gamma(\psi)] g(x) - [f''(x) + g''(x)\gamma(\psi)] g(x) \right. \\ &\quad \left. - 2 [f'(x) + g'(x)\gamma(\psi)] g'(x) - 2 [f'(x) + g'(x)\gamma(\psi)]^3 g'(x) \right\} \gamma'^2(\psi) \\ &\quad + \left\{ 1 + [f'(x) + g'(x)\gamma(\psi)]^2 \right\}^2 \gamma''(\psi) = 0\end{aligned}\tag{6.30}$$

We study the following five cases separately:

- (1) $g'(x) = 0$,
- (2) $f(x) = 0$,
- (3) $f''(x) = 0$, $g''(\psi) = 0$
- (4) $\gamma''(\psi) = 0$, $g'(x) \neq 0$, $f(x) \neq 0$
- (5) $f(x) \neq 0$, $g'(x) \neq 0$, $\gamma''(\psi) \neq 0$

6.4.1 Solutions for $g'(x) = 0$.

Without loss of generality we assume that $g(x) = 1$. With $g(x) = 1$ equation (6.30) reduces to

$$[f'^2(x) - 1] f''(x) \gamma'^2(\psi) + [1 + f'^2(x)]^2 \gamma''(\psi) = 0\tag{6.31}$$

Dividing equation (6.31) by $[1 + f'^2(x)]^2 \gamma'^2(\psi) \neq 0$, we obtain

$$\frac{[f'^2(x) - 1] f''(x)}{[1 + f'^2(x)]^2} + \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0$$

This equation implies that

$$\begin{aligned}\frac{[f'^2(x) - 1] f''(x)}{[1 + f'^2(x)]^2} &= a_1 \\ \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= -a_1\end{aligned}\tag{6.32}$$

where a_1 is an arbitrary constant.

Case (a): $a_1 = 0$. If $a_1 = 0$, the equations (6.32) become

$$f''(x) = 0 \quad \text{and} \quad \gamma''(\psi) = 0$$

which upon integration gives

$$f(x) = a_2 x + a_3\tag{6.33}$$

$$\gamma(\psi) = a_4 \psi + a_5$$

where $a_2, a_3, a_4 \neq 0$ and a_5 are arbitrary constants of integration. Employing $g(x) = 1$, (6.27), (6.29) and (6.33) in equations (6.21) and (6.24), we find that

$$\omega = 0, \quad \Omega = 0$$

Using $\omega = \Omega = 0$, equation (6.22) is identically satisfied. Thus, the family of curves $y - a_2 x - a_3 = \text{constant}$ is a possible streamline pattern for infinitely conducting irrotational MHD orthogonal flow.

Case (b): $a_1 \neq 0$. Integrating equations (6.32), we obtain

$$\begin{aligned}f(x) = \frac{1}{2a_1} [\ln(a_1 x + a_6) \mp \ln(a_1 x + a_6)] \pm \frac{1}{2a_1} \left[\sqrt{1 - 4(a_1 x + a_6)^2} \right. \\ \left. - \ln \left(\frac{1}{2} + \frac{1}{2} \sqrt{1 - 4(a_1 x + a_6)^2} \right) + a_7 \right]\end{aligned}\tag{6.34}$$

$$\gamma(\psi) = \frac{1}{a_1} \ln(a_1 \psi + a_8) + a_9$$

where a_6 to a_9 are arbitrary constants of integration. Using (6.27), (6.28), (6.29) and (6.34) in equations (6.21) and (6.24), we get

$$\begin{aligned}\omega &= \mp \frac{2a_1(a_1 \psi + a_8)}{\sqrt{1 - 4(a_1 x + a_6)^2}} \\ \Omega &= \mp \frac{2Ca_1(a_1 x + a_6)}{\sqrt{1 - 4(a_1 x + a_6)^2}} \mp \frac{1}{a_1 \psi + a_8}\end{aligned}\tag{6.35}$$

Using (6.27), (6.29) and (6.34) in equation (6.22), we find that $a_1 = 0$. This contradicts our assumption that $a_1 \neq 0$. Summing up, we have:

Theorem 6.2. *The only possible streamline pattern for infinitely conducting MHD orthogonal flow that belongs to the form $y - f(x) = \text{constant}$ is $y - a_2x = \text{constant}$ and this flow is irrotational.*

6.4.2. Solutions for $f(x) = 0$.

If $f(x) = 0$, then equation (6.30) gives

$$\begin{aligned} & \{ [g(x)g''(x) - 2g'^2(x)] g'^2(x)\gamma^3(\psi) - [g(x)g''(x) + 2g'^2(x)] \gamma(\psi) \} \gamma'^2(\psi) \\ & + [1 + g'^2(x)\gamma^2(\psi)]^2 \gamma''(\psi) = 0 \end{aligned} \quad (6.36)$$

Dividing equation (6.36) by $\gamma(\psi)\gamma'^2(\psi) \neq 0$ and differentiating the resulting equation with respect to ψ , we obtain

$$\begin{aligned} & 2 [g(x)g'^2(x)g''(x) - 2g'^4(x)] \gamma(\psi)\gamma'(\psi) + \left(\frac{\gamma''(\psi)}{\gamma(\psi)\gamma'^2(\psi)} \right)' \\ & + g'^4(x) \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + 2g'^2(x) \left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' = 0 \end{aligned} \quad (6.37)$$

Dividing equation (6.37) by $\gamma(\psi)\gamma'(\psi) \neq 0$ and differentiating the resulting equation once with respect to ψ and then with respect to x , we have

$$\left[g'^2(x) \left\{ \frac{1}{\gamma(\psi)\gamma'(\psi)} \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\} + \left\{ \frac{1}{\gamma(\psi)\gamma'(\psi)} \left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\} \right] g''(x) = 0 \quad (6.38)$$

Equation (6.38) implies that either $g''(x) = 0$ or $g''(x) \neq 0$. So, we have the following two subcases:

Case (i): $g''(x) = 0$. Integrating $g''(x) = 0$ twice with respect to x , we get

$$g(x) = c_1x + c_2 \quad (6.39)$$

where c_1 and c_2 are arbitrary constants of integration such that are not equal to zero simultaneously.

Using (6.39) in equation (6.36), we obtain

$$[1 + c_1^2 \gamma^2(\psi)] \gamma''(\psi) - 2c_1^2 \gamma(\psi) \gamma'^2(\psi) = 0 \quad (6.40)$$

which upon integration implies that

$$\gamma(\psi) = \frac{1}{c_1} \tan [c_1 (c_3 \psi + c_4)] \quad (6.41)$$

where $c_3 \neq 0$ and c_4 are arbitrary constants and we assume that $c_1 \neq 0$. Employing (6.27) to (6.29) and (6.41) in (6.21) and (6.24), we get

$$\begin{aligned} \omega &= 0 \\ \Omega &= 2C c_1 c_3 \end{aligned} \quad (6.42)$$

Substituting (6.27), (6.29) and (6.42), equation (6.22) is identically satisfied.

If $c_1 = 0$, then equation (6.40) yields

$$\gamma''(\psi) = 0$$

which upon integration gives

$$\gamma(\psi) = c_5 \psi + c_6 \quad (6.43)$$

where $c_5 \neq 0$ and c_6 are arbitrary constants. Using (6.21), (6.24), (6.27), (6.29) and (6.43) in (6.42), we get $c_5 = 0$ which is not possible.

Case (ii): $g''(x) \neq 0$. If $g''(x) \neq 0$, then equation (6.38) gives

$$g'^2(x) \left\{ \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma^3(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\}' + \left\{ \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\}' = 0 \quad (6.44)$$

If $\left\{ \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\}' \neq 0$, then equation (6.44) implies that $g'^2(x) =$ constant or $g''(x) = 0$ which contradicts our assumption that $g''(x) \neq 0$. Thus, we have

$$\left\{ \frac{1}{\gamma(\psi) \gamma'(\psi)} \left(\frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\}' = 0 \quad (6.45)$$

Employing (6.45) in equation (6.44), we get

$$\left\{ \frac{1}{\gamma(\psi)\gamma'(\psi)} \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right\}' = 0 \quad (6.46)$$

since $g'^2(x) \neq 0$.

Integrating equation (6.45) twice with respect to ψ , we obtain

$$\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} = \frac{1}{2}b_1\gamma^2(\psi) + b_2 \quad (6.47)$$

Using (6.47) in equation (6.46), we get

$$b_1 = 0$$

Employing (6.47) with $b_1 = 0$, equation (6.37) gives

$$[g(x)g'^2(x)g''(x) + (b_2 - 2)g'^4(x)]\gamma^4(\psi) - b_2 = 0$$

This equation implies that

$$\begin{aligned} b_2 &= 0 \\ g(x)g''(x) - 2g'^2(x) &= 0 \end{aligned} \quad (6.48)$$

With $b_1 = b_2 = 0$, equation (6.47) gives

$$\gamma''(\psi) = 0 \quad (6.49)$$

Using (6.48) and (6.49), in (6.36), we get

$$g(x)g''(x) + 2g'^2(x) = 0 \quad (6.50)$$

Using (6.50) in (6.48), we get

$$g'(x) = 0$$

which contradicts our assumption that $g''(x) \neq 0$. Summing up, we have the following theorem:

Theorem 6.3. *The only possible streamline pattern for infinitely conducting MHD orthogonal flow that belongs to the form $\frac{y}{g(x)} = \text{constant}$ is $\frac{y}{c_1 x + c_2} = \text{constant}$ and this flow is irrotational.*

6.4.3. Solutions for $f''(x) = 0, g''(x) = 0$.

Integrating $f''(x) = 0$ and $g''(x) = 0$ twice with respect to x , we obtain

$$\begin{aligned} f(x) &= a_1 x + a_4 \\ g(x) &= a_3 x + a_4 \end{aligned} \tag{6.51}$$

where a_1 to a_4 are arbitrary constants such that $a_3 \neq 0$.

Using equation (6.51) in (6.30) and integrating the resulting equation twice with respect to ψ , we get

$$\gamma(\psi) = \frac{1}{a_3} [\tan(a_3 a_5 \psi + a_3 a_6) - a_1] \tag{6.52}$$

Employing (6.27), (6.29), (6.51) and (6.52) in equations (6.21) and (6.24), we find that

$$\omega = 0, \quad \Omega = 2a_3 C \tag{6.53}$$

Substituting (6.27), (6.29) and (6.51) to (6.53) into equation (6.22), we find that this equation is identically satisfied. Thus, we have the following theorem:

Theorem 6.4. *Steady plane flow along $\frac{y - (a_1 x + a_2)}{a_3 x + a_4} = \text{constant}$ is permissible for infinitely conducting irrotational MHD orthogonal flow.*

6.4.4. Solutions for $\gamma''(\psi) = 0, g'(x) \neq 0, f(x) \neq 0$.

Integrating $\gamma''(\psi) = 0$, we obtain

$$\gamma(\psi) = a_1 \psi + a_2 \tag{6.54}$$

where $a_1 \neq 0$ and a_2 are arbitrary constants. Using (6.54) in equation (6.30), we have

$$\begin{aligned} & [g(x)g'^2(x)g''(x) - 2g'^4(x)](a_1\psi + a_2)^3 + [f''(x)g(x)g'^2(x) - 6f'(x)g'^3(x) \\ & + 2f'(x)g(x)g'(x)g''(x)](a_1\psi + a_2)^2 + [f'^2(x)g(x)g''(x) - g(x)g''(x) \\ & + 2f'(x)f''(x)g(x)g'(x) - 2g'^2(x) - 6f'^2(x)g'^2(x)](a_1\psi + a_2) \\ & + [f'^2(x)f''(x)g(x) - f''(x)g(x) - 2f'(x)g'(x) - 2f'^3(x)g'(x)] = 0 \end{aligned}$$

This equation is a third degree polynomial in $(a_1\psi + a_2)$ with coefficients as functions of x only. For this polynomial to hold true for all values of ψ , the coefficients of different power of $(a_1\psi + a_2)$ must vanish simultaneously. This requirement gives

$$g(x)g'^2(x)g''(x) - 2g'^4(x) = 0 \quad (6.55)$$

$$f''(x)g(x)g'^2(x) - 6f'(x)g'^3(x) + 2f'(x)g(x)g'(x)g''(x) = 0 \quad (6.56)$$

$$f'^2(x)g(x)g''(x) - g(x)g''(x) + 2f'(x)f''(x)g(x)g'(x) \quad (6.57)$$

$$- 2g'^2(x) - 6f'^2(x)g'^2(x) = 0$$

$$f'^2(x)f''(x)g(x) - f''(x)g(x) - 2f'(x)g'(x) - 2f'^3(x)g'(x) = 0 \quad (6.58)$$

Integrating equation (6.55), we obtain

$$g(x) = -\frac{1}{a_3x + a_4} \quad (6.59)$$

where $a_3 \neq 0$ and a_4 are arbitrary constants of integration.

Using (6.59) in (6.56) and integrating the resulting equation, we have

$$f(x) = -\frac{a_5}{a_3} \frac{1}{a_3x + a_4} + a_6 \quad (6.60)$$

where a_5 and a_6 are arbitrary constants. Employing (6.59) and (6.60) in (6.57), we find that $a_3 = 0$, which contradicts our assumption that $a_3 \neq 0$.

Suppose that $a_3 = 0$, then equation (6.58) implies that $f''(x) = 0$. But the case where $\gamma''(\psi) = f''(x) = 0$, was considered in case (a) of Section 6.3.1 above.

6.4.5. Solutions for $f''(x) \neq 0$, $g''(x) \neq 0$, $\gamma''(\psi) \neq 0$.

Rewriting equation (6.30), we get

$$\begin{aligned} A_0(x) + A_1(x)\gamma(\psi) + A_2(x)\gamma^2(\psi) + A_3(x)\gamma^3(\psi) + A_4(x)\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \\ + A_5(x)\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} + A_6(x)\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} + A_7(x)\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \\ + A_8(x)\frac{\gamma^4(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} = 0 \end{aligned} \quad (6.61)$$

where

$$\begin{aligned} A_0(x) &= [f'^2(x) - 1] f''(x)g(x) - 2f'(x) [1 + f'^2(x)] g'(x) \\ A_1(x) &= 2f'(x)f''(x)g(x)g'(x) + [f'^2(x) - 1] g(x)g''(x) - 2 [1 + 3f'^2(x)] g'^2(x) \\ A_2(x) &= 2f'(x)g(x)g'(x)g''(x) - 6f'(x)g'^3(x) + f''(x)g(x)g'^2(x) \\ A_3(x) &= g(x)g'^2(x)g''(x) - 2g'^4(x) \\ A_4(x) &= [1 + f'^2(x)]^2 \\ A_5(x) &= 4f'(x)g'(x) [1 + f'^2(x)] \\ A_6(x) &= 2g'^2(x) [1 + 3f'^2(x)] \\ A_7(x) &= 4f'(x)g'^3(x) \\ A_8(x) &= g'^4(x) \end{aligned}$$

Differentiating equation (5.61) with respect to ψ , we have

$$\begin{aligned} A_1(x)\gamma'(\psi) + 2A_2(x)\gamma(\psi)\gamma'(\psi) + 3A_3(x)\gamma^2(\psi)\gamma'(\psi) + A_4(x)\left(\frac{\gamma''(\psi)}{\gamma'(\psi)}\right)' \\ + A_5(x)\left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right) + A_6(x)\left(\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right) + A_7(x)\left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right) \\ + A_8(x)\left(\frac{\gamma^4(\psi)\gamma''(\psi)}{\gamma'^2(\psi)}\right) = 0 \end{aligned} \quad (6.62)$$

Dividing equation (5.62) by $\gamma'(\psi) \neq 0$ and differentiating the resulting equation

once again with respect to ψ , we obtain

$$\begin{aligned}
& 2A_2\gamma'(\psi) + 6A_3(x)\gamma(\psi)\gamma'(\psi) + A_4(x) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \\
& + A_5(x) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' + A_6(x) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \\
& + A_7(x) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' + A_8(x) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^4(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' = 0
\end{aligned} \tag{6.63}$$

Dividing equation (6.63) by $\gamma'(\psi) \neq 0$ and differentiating the resulting equation with respect to ψ , we get

$$\begin{aligned}
& 6A_3(x)\gamma'(\psi) + A_4(x) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \\
& + A_5(x) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \\
& + A_6(x) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \\
& + A_7(x) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \\
& + A_8(x) \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^4(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' = 0
\end{aligned} \tag{6.64}$$

Dividing equation (6.64) by $\gamma'(\psi) \neq 0$ and differentiating the resulting equation with respect to ψ , we have

$$A_4(x)B_0(\psi) + A_5(x)B_1(\psi) + A_6(x)B_2(\psi) + A_7(x)B_3(\psi) + A_8(x)B_4(\psi) = 0 \tag{6.65}$$

where

$$\begin{aligned}
B_0(\psi) &= \left(\frac{1}{\gamma'(\psi)} \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \right)' \\
B_1(\psi) &= \left(\frac{1}{\gamma'(\psi)} \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \right)' \\
B_2(\psi) &= \left(\frac{1}{\gamma'(\psi)} \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^2(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \right)' \\
B_3(\psi) &= \left(\frac{1}{\gamma'(\psi)} \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^3(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \right)' \\
B_4(\psi) &= \left(\frac{1}{\gamma'(\psi)} \left\{ \frac{1}{\gamma'(\psi)} \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma^4(\psi)\gamma''(\psi)}{\gamma'^2(\psi)} \right)' \right]' \right\}' \right)'
\end{aligned}$$

Dividing (6.65) by $A_8(x) = g'^4(x) \neq 0$ and differentiating the resulting equation with respect to x , we obtain

$$B_0(\psi) \left(\frac{A_4(x)}{A_8(x)} \right)' + B_1(\psi) \left(\frac{A_5(x)}{A_8(x)} \right)' + B_2(\psi) \left(\frac{A_6(x)}{A_8(x)} \right)' + B_3(\psi) \left(\frac{A_7(x)}{A_8(x)} \right)' = 0 \quad (6.66)$$

We consider the following two cases:

$$(a) \left(\frac{A_7(x)}{A_8(x)} \right)' = 0$$

$$(b) \left(\frac{A_7(x)}{A_8(x)} \right)' \neq 0$$

Case (a): $\left(\frac{A_7(x)}{A_8(x)} \right)' = 0$. Integrating $\left(\frac{A_7(x)}{A_8(x)} \right)' = 0$ twice with respect to x , we have

$$f(x) = a_1 g(x) + a_2 \quad (6.67)$$

where a_1 and a_2 are arbitrary constants. Using (6.67), equation (6.66) yields

$$-\frac{4g''(x)}{g'^3(x)} \left\{ \left(\frac{1}{g'^2(x)} + 1 \right) B_0(\psi) - 2a_1 B_1(\psi) + B_2(\psi) \right\} = 0 \quad (6.68)$$

Equation (6.66) implies that either

$$g''(x) = 0, \quad \left(\frac{1}{g'^2(x)} + 1 \right) B_0(\psi) - 2a_1 B_1(\psi) + B_2(\psi) \neq 0$$

or

$$\left(\frac{1}{g'^2(x)} + 1 \right) B_0(\psi) - 2a_1 B_1(\psi) + B_2(\psi) = 0, \quad g''(x) \neq 0$$

Subcase (i): $g''(x) = 0$. Integrating $g''(x) = 0$ twice with respect to x , we have

$$g(x) = a_3 x + a_4 \quad (6.69)$$

where $a_3 \neq 0$ and a_4 are arbitrary constants. Employing (6.67) and (6.69), equation (6.61) gives

$$\begin{aligned} & -2a_1 a_3^2 [1 + a_1^2 a_3^2] - 2 [1 + 3a_1^2 a_3^2] a_3^2 \gamma(\psi) - 6a_1 a_3^4 \gamma^2(\psi) - 2a_3^4 \gamma^3(\psi) \\ & + [1 + a_1^2 a_3^2]^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} + 4a_1 a_3^2 [1 + a_1^2 a_3^2] \frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} \\ & + 2a_3 [1 + 3a_1^2 a_3^2] \frac{\gamma^2(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} + 4a_1 a_3^4 \frac{\gamma^3(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} + a_3^4 \frac{\gamma^4(\psi) \gamma''(\psi)}{\gamma'^2(\psi)} = 0 \end{aligned} \quad (6.70)$$

Using (6.21), (6.24), (6.27) to (6.29), (6.67) and (6.69) in equation (6.22), we obtain

$$\begin{aligned} & 6a_3^2 L(\psi) + 4a_3^2 [a_1 + \gamma(\psi)] L'(\psi) + \left\{ 1 + a_3^2 [a_1 + \gamma(\psi)]^2 \right\} \frac{1}{\gamma'(\psi)} L''(\psi) \\ & - \gamma'(\psi) L(\psi) L'(\psi) + \frac{2a_3 \rho}{\mu} L(\psi) + \frac{2C^2 a_3 \mu^*}{\mu} \left\{ \frac{2\gamma'(\psi) \gamma''(\psi)}{1 + a_3^2 [a_1 + \gamma(\psi)]^2} \right. \\ & \left. - \frac{4a_3^2 [a_1 + \gamma(\psi)] \gamma'^3(\psi)}{\left\{ 1 + a_3^2 [a_1 + \gamma(\psi)]^2 \right\}^2} \right\} (a_3 x + a_4)^4 = 0 \end{aligned} \quad (6.71)$$

where

$$\begin{aligned} L(\psi) = & -2a_1 a_3^2 \frac{1}{\gamma'(\psi)} - 2a_3^2 \frac{\gamma(\psi)}{\gamma'(\psi)} + (1 + a_1^2 a_3^2) \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + 2a_1 a_3^2 \frac{\gamma(\psi) \gamma''(\psi)}{\gamma'^3(\psi)} \\ & + a_3^2 \frac{\gamma^2(\psi) \gamma''(\psi)}{\gamma'^3(\psi)} \end{aligned}$$

Equation (6.71) is a fourth degree polynomial in $(a_3x + a_4)$ with coefficients as functions of ψ only. Requiring the coefficient of $(a_3x + a_4)^4$ to be zero, we obtain after integrating twice with respect to ψ

$$\gamma(\psi) = \frac{1}{a_3} \tan(a_3a_5\psi + a_3a_6) - a_1 \quad (6.72)$$

where $a_5 \neq 0$ and a_6 are arbitrary constants. Using equation (6.72) in (6.70), we get $a_1 = 0$ which implies that $\omega = 0$. Thus $\frac{y - a_2}{a_3x + a_4} = \text{constant}$ is a possible streamline pattern for infinitely conducting irrotational MHD orthogonal flow.

Subcase (ii): $g''(x) \neq 0$. If $g''(x) \neq 0$, then equation (6.68) gives

$$\left\{ \left(\frac{1}{g'^2(x)} + 1 \right) B_0(\psi) - 2a_1B_1(\psi) + B_2(\psi) \right\} = 0 \quad (6.73)$$

Differentiating (6.73) with respect to x , we get

$$-\frac{2g''(x)}{g'^3(x)} B_0(\psi) = 0$$

which implies that

$$B_0(\psi) = 0$$

since $g''(x) \neq 0$. Integrating $B_0(\psi) = 0$ four times with respect to ψ , we obtain

$$\frac{\gamma''(\psi)}{\gamma'^2(\psi)} = c_1\gamma^3(\psi) + c_2\gamma^2(\psi) + 2c_3\gamma(\psi) + c_4 \quad (6.74)$$

where c_1 to c_4 are arbitrary constants such that c_1 to c_3 are not simultaneously zero.

Using (6.74) in (6.73), we have

$$c_1 = c_2 = 0$$

Employing $c_1 = c_2 = 0$ in (6.65), we find that

$$c_3 = c_4 = 0$$

With $c_1 = c_2 = c_3 = c_4 = 0$, equation (6.74) implies that $\gamma''(\psi) = 0$ which contradicts our assumption that $\gamma''(\psi) \neq 0$.

Case (b): $\left(\frac{A_7(x)}{A_8(x)}\right)' \neq 0$. Dividing equation (6.66) by $\left(\frac{A_7(x)}{A_8(x)}\right)' \neq 0$ and differentiating the resulting equation with respect to x , we get

$$B_0(\psi) \left[\frac{\left(\frac{A_4(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' + B_1(\psi) \left[\frac{\left(\frac{A_5(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' + B_2(\psi) \left[\frac{\left(\frac{A_6(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' = 0 \quad (6.75)$$

We shall consider the following two subcases

$$(i) B_0(\psi) = 0$$

$$(ii) B_0(\psi) \neq 0$$

Subcase (i): $B_0(\psi) = 0$. Integrating $B_0(\psi) = 0$ four times with respect to ψ , we obtain

$$\frac{\gamma''(\psi)}{\gamma'^2(\psi)} = \lambda_1 \gamma^3(\psi) + \lambda_2 \gamma^2(\psi) + \lambda_3 \gamma(\psi) + \lambda_4 \quad (6.76)$$

where λ_1 to λ_4 are arbitrary constants such that λ_1 to λ_3 are not zero at the same time. Using (6.76), equation (6.75) yields

$$24\lambda_1 \left[\frac{\left(\frac{A_5(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' + [120\lambda_1 \gamma(\psi) + 24\lambda_2] \left[\frac{\left(\frac{A_6(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' = 0 \quad (6.77)$$

Differentiating equation (6.77) with respect to ψ , we have

$$120\lambda_1 \left[\frac{\left(\frac{A_6(x)}{A_8(x)}\right)'}{\left(\frac{A_7(x)}{A_8(x)}\right)'} \right]' = 0 \quad (6.78)$$

If $\lambda_1 = 0$, then equation (6.77) gives

$$\lambda_2 = 0$$

Employing (6.76) with $\lambda_1 = \lambda_2 = 0$ in equation (6.66), we find that

$$\lambda_3 = \lambda_4 = 0$$

which implies that $\gamma''(\psi) = 0$. This is not possible because we have assumed that $\gamma''(\psi) \neq 0$. Thus, we have to assume that $\lambda_1 \neq 0$.

Integrating (6.78) once with respect to x , we get

$$\left(\frac{A_6(x)}{A_8(x)} \right)' = \lambda_5 \left(\frac{A_7(x)}{A_8(x)} \right)' \quad (6.79)$$

Using (6.78) in equation (6.77) and integrating the resulting equation, we obtain

$$\left(\frac{A_5(x)}{A_8(x)} \right)' = \lambda_6 \left(\frac{A_7(x)}{A_8(x)} \right)' \quad (6.80)$$

Substituting (6.76), (6.79) and (6.80) in equation (6.66), we have

$$24\lambda_1\lambda_6 + \lambda_5 [120\lambda_1\gamma(\psi) + 24\lambda_2] + 360\lambda_1\gamma^2(\psi) + 120\lambda_2\gamma(\psi) + 24\lambda_3 = 0$$

which implies that $\lambda_1 = 0$ contradicting the assumption that $\lambda_1 \neq 0$. Hence, $B_0(\psi)$ cannot be zero.

Subcase (ii): $B_0(\psi) \neq 0$. Dividing equation (6.75) by $B_0(\psi) \neq 0$ and differentiating the resulting equation with respect to ψ , we obtain

$$\left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' \left[\frac{B_1(\psi)}{B_0(\psi)} \right]' + \left[\frac{\left(\frac{A_6(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' = 0 \quad (6.81)$$

We shall consider the following four subcases:

$$\begin{aligned}
 (1). \quad & \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' = 0, & \left[\frac{B_1(\psi)}{B_0(\psi)} \right]' &= 0 \\
 (2). \quad & \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' = 0, & \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' &= 0 \\
 (3). \quad & \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' = 0, & \left[\frac{\left(\frac{A_6(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' &= 0 \\
 (4). \quad & \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' \neq 0, & \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' &\neq 0
 \end{aligned}$$

Subsubcase (1): $\left\{ \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' = 0, \quad \left[\frac{B_1(\psi)}{B_0(\psi)} \right]' = 0 \right\}$. Integrating these two equations once with respect to ψ , we obtain

$$\begin{aligned}
 B_2(\psi) &= m_1 B_0(\psi) \\
 B_1(\psi) &= m_2 B_0(\psi)
 \end{aligned} \tag{6.82}$$

where m_1 and m_2 are arbitrary constants.

Using equations (6.82) in (6.73) and integrating the resulting equation twice with respect to x , we have

$$A_4(x) + m_2 A_5(x) + m_1 A_6(x) - m_3 A_7(x) - m_4 A_8(x) = 0 \tag{6.83}$$

where m_3 and m_4 are arbitrary constants of integration. Employing (6.82) and (6.83), equation (6.66) yields

$$B_3(\psi) = -m_3 B_0(\psi) \tag{6.84}$$

Substituting (6.82) to (6.84) in equation (6.65), we get

$$B_4(\psi) = -m_4 B_0(\psi) \tag{6.85}$$

Integrating (6.82), (6.84) and (6.85) four times with respect to ψ , we obtain

$$\begin{aligned}
[\gamma^2(\psi) - m_1] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= m_5\gamma^3(\psi) + m_6\gamma^2(\psi) + m_7\gamma(\psi) + m_8 \\
[\gamma(\psi) - m_2] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= m_9\gamma^3(\psi) + m_{10}\gamma^2(\psi) + m_{11}\gamma(\psi) + m_{12} \\
[\gamma^3(\psi) + m_3] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= m_{13}\gamma^3(\psi) + m_{14}\gamma^2(\psi) + m_{15}\gamma(\psi) + m_{16} \\
[\gamma^4(\psi) + m_4] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= m_{17}\gamma^3(\psi) + m_{18}\gamma^2(\psi) + m_{19}\gamma(\psi) + m_{20}
\end{aligned} \tag{6.86a,b,c,d}$$

Dividing (6.86d) by (6.86b), we get

$$\begin{aligned}
&m_9\gamma^7(\psi) + m_{10}\gamma^6(\psi) + m_{11}\gamma^5(\psi) + [m_{12} - m_{17}]\gamma^4(\psi) \\
&\quad + [m_4m_9 - m_{13} + m_2m_{17}]\gamma^3(\psi) + [m_4m_{10} - m_{19} + m_2m_{18}]\gamma^2(\psi) \\
&\quad + [m_4m_{11} - m_{20} + m_2m_{19}]\gamma(\psi) + m_4m_{12} + m_2m_{20} = 0
\end{aligned} \tag{6.87}$$

Equation (6.87) is a seventh degree polynomial in $\gamma(\psi)$. Since this polynomial must hold true for all values of ψ , we have

$$\begin{aligned}
m_9 &= m_{10} = m_{11} = 0 \\
m_{12} - m_{17} &= 0 \\
m_4m_9 - m_{13} + m_2m_{17} &= 0 \\
m_4m_{10} - m_{19} + m_2m_{18} &= 0 \\
m_4m_{11} - m_{20} + m_2m_{19} &= 0 \\
m_4m_{12} + m_2m_{20} &= 0
\end{aligned} \tag{6.88}$$

Dividing (6.86a) by (6.86b) with $m_9 = m_{10} = m_{11} = 0$, we obtain

$$\begin{aligned}
&m_5\gamma^4(\psi) + [m_6 - m_5m_2]\gamma^3(\psi) + [m_7 - m_2m_6 - m_{12}]\gamma^2(\psi) \\
&\quad + [m_8 - m_2m_7]\gamma(\psi) + m_1m_{12} - m_2m_8 = 0
\end{aligned}$$

This equation gives

$$\begin{aligned}
m_5 &= m_6 = 0 \\
m_7 - m_2m_6 - m_{12} &= 0 \\
m_8 - m_2m_7 &= 0 \\
m_1m_{12} - m_2m_8 &= 0
\end{aligned} \tag{6.89}$$

Dividing (6.86c) by (6.86b) with $m_9 = m_{10} = m_{11} = 0$, we have

$$\begin{aligned} m_{13}\gamma^4(\psi) + [m_{14} - m_2m_{13} - m_{12}]\gamma^3(\psi) + [m_{15} - m_2m_{14}]\gamma^2(\psi) \\ + [m_{16} - m_2m_{15}]\gamma(\psi) - m_2m_{16} - m_3m_{12} = 0 \end{aligned}$$

This equation implies that

$$\begin{aligned} m_{13} &= 0 \\ m_{14} - m_2m_{13} - m_{12} &= 0 \\ m_{15} - m_2m_{14} &= 0 \\ m_{16} - m_2m_{15} &= 0 \\ m_2m_{16} + m_3m_{12} &= 0 \end{aligned} \tag{6.90}$$

If $m_{12} = 0$, then equation (6.84b) implies that $\gamma''(\psi) = 0$ since $m_9 = m_{10} = m_{11} = 0$. We assume that $m_{12} \neq 0$. Using (6.88) to (6.90), equations (6.86) reduces to

$$[\gamma(\psi) - m_2] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = m_{12} \tag{6.91}$$

Employing (6.91) in (6.64), we obtain

$$\begin{aligned} [A_3(x) + m_{12}A_8(x)]\gamma^4(\psi) - 4m_2A_3(x)\gamma^3(\psi) + [6m_2^2A_3(x) + 2m_2^2m_{12}A_8(x)]\gamma^2(\psi) \\ - [4m_2^3A_3(x) + 4m_2^3m_{12}A_8(x)]\gamma(\psi) + m_2^4A_3(x) - m_{12}A_4(x) - m_2m_{12}A_5(x) \\ - m_2^2m_{12}A_6(x) - m_2^3m_{12}A_7(x) = 0 \end{aligned}$$

This equation is a fourth degree polynomial in $\gamma(\psi)$ with coefficients as functions of x only. Thus, we have

$$\begin{aligned} A_3(x) + m_{12}A_8(x) &= 0 \\ -4m_2A_3(x) &= 0 \\ 6m_2^2A_3(x) + 2m_2^2m_{12}A_8(x) &= 0 \\ -4m_2^3A_3(x) - 4m_2^3m_{12}A_8(x) &= 0 \\ m_2^4A_3(x) - m_{12}A_4(x) - m_2m_{12}A_5(x) - m_2^2m_{12}A_6(x) - m_2^3m_{12}A_7(x) &= 0 \end{aligned} \tag{6.92a,b,c,d,e}$$

Equation (6.92b) implies that either $m_2 = 0$ or $A_3(x) = 0$. If $m_2 = 0$, then equation (6.92e) implies that $A_4(x) = [1 + f'^2(x)]^2 = 0$ which is impossible. If $A_3(x) = 0$, then equation (6.92a) implies that $A_8(x) = g'^4(x) = 0$ which contradicts our assumption that $g'(x) \neq 0$. Hence, we conclude that subsubcase (1) is not possible.

Subsubcase (2) : $\left\{ \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' = 0, \quad \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' = 0 \right\}$. Integrating the first equation with respect to ψ and the second equation twice with respect to x , we obtain

$$B_2(\psi) = n_1 B_0(\psi) \quad (6.93a,b)$$

$$A_5(x) = n_2 A_7(x) + n_3 A_8(x)$$

where n_1 to n_3 are arbitrary constants. Employing equations (6.93), equation (6.75) after two integrations yields

$$A_4(x) + n_1 A_6(x) - n_4 A_7(x) - n_5 A_8(x) = 0 \quad (6.94)$$

where n_4 and n_5 are arbitrary constants. Substituting (6.93) and (6.94) in equation (6.66), we get

$$B_3(\psi) = -n_2 B_1(\psi) - n_4 B_0(\psi) \quad (6.95)$$

Using (6.93) to (6.95), equation (6.65) gives

$$B_4(\psi) = -n_5 B_0(\psi) - n_3 B_1(\psi) \quad (6.96)$$

Integrating equations (6.93a), (6.95) and (6.96) four times with respect to ψ , we have

$$\begin{aligned} [\gamma^2(\psi) - n_1] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= n_6 \gamma^3(\psi) + n_7 \gamma^2(\psi) + n_8 \gamma(\psi) + n_9 \\ [\gamma^3(\psi) + n_2 \gamma(\psi) + n_4] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= n_{10} \gamma^3(\psi) + n_{11} \gamma^2(\psi) + n_{12} \gamma(\psi) + n_{13} \\ [\gamma^4(\psi) + n_3 \gamma(\psi) + n_5] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= n_{14} \gamma^3(\psi) + n_{15} \gamma^2(\psi) + n_{16} \gamma(\psi) + n_{17} \end{aligned} \quad (6.97a,b,c)$$

Dividing (6.97c) by (6.97a), (6.97c) by (6.97b) and (6.97b) by (6.97a) and equating coefficients of different powers of $\gamma(\psi)$ to zero, we get

$$\begin{aligned}
n_8 = n_7 = n_{10} = 0, \quad n_{11} = n_8, \quad n_{12} = n_9, \quad n_{13} = (n_1 + n_2)n_8, \\
n_{14} = n_8, \quad n_{15} = n_9, \quad n_{16} = n_1n_8, \quad n_{17} = n_3n_8 + n_1n_9 \\
n_5n_9 + n_1(n_3n_8 + n_1n_9) = 0, \quad n_5(n_1 + n_2)n_8 - n_4(n_3n_8 + n_1n_9) = 0 \\
(n_1 + n_2)n_9 + n_4n_8 = 0, \quad (n_5 - n_1n_2)n_9 - (n_2n_3 + n_4)n_8 = 0 \\
n_1(n_3 - n_4)n_8 + (n_5 - n_1n_2)n_9 = 0, \quad (n_5 + n_1^2)n_8 + n_3n_9 = 0 \\
(n_3 - n_4)n_9 + (n_5 - n_1n_2)n_8 = 0, \quad n_4n_9 - n_1(n_1 + n_2)n_8 = 0
\end{aligned} \tag{6.98}$$

Using (6.98), equations (6.97) reduces to

$$[\gamma^2(\psi) - n_1] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = n_8\gamma(\psi) + n_9$$

The last three equations in system (6.98) yields

$$n_1(n_1 + n_2)n_8 = 0$$

If we take $n_1 = 0$ in (6.98), then either $n_9 = n_8 = 0$ which implies that $\gamma''(\psi) = 0$ or $n_5 = n_4 = 0$. Using $n_1 = n_4 = n_5 = 0$ in equation (6.94), we get $A_4(x) = [1 + f'^2(x)]^2 = 0$. Thus, we assume that $n_1 \neq 0$.

When $n_8 = 0$ and $n_1 + n_2 \neq 0$ are taken in (6.98), we get $n_3 = n_4 = 0$ and $n_1 + n_2 = 0$ which contradicts our assumption that $n_1 + n_2 \neq 0$. If $n_8 = 0$ and $n_1 + n_2 = 0$ are taken in system (6.98), we obtain $n_3 = n_4 = n_1 = 0$. But we showed above that $n_1 \neq 0$.

Thus, we only need to consider the case where $n_1 + n_2 = 0$ and $n_8 \neq 0$. When $n_1 + n_2 = 0$ is taken, system (6.98) gives

$$n_4 = n_{13} = 0$$

Using equations (6.93) to (6.96) in (6.65) and (6.63), we get

$$\begin{aligned}
A_3(x) + n_8A_8(x) &= 0 \\
A_2(x) + n_9A_8(x) &= 0
\end{aligned} \tag{6.99a,b}$$

Integrating (6.99a) twice with respect to x , we get

$$g(x) = \begin{cases} e^{a_1 x + a_2}, & n_8 = 1 \\ a_3 x + a_4, & n_8 = 2 \\ [n_8 - 1]^{\frac{1}{n_8-1}} [a_1 x + a_5]^{\frac{1}{n_8-1}}, & n_8 \neq 1, n_8 \neq 2 \end{cases} \quad (6.100)$$

where a_1 to a_5 are arbitrary constants of integration. Employing (6.100) in (6.99b),

and integrating the resulting equation, we obtain

$$f(x) = \begin{cases} \frac{n_9}{3} e^{a_1 x + a_2} + \frac{a_6}{4a_1^3} e^{4(a_1 x + a_2)} + a_7, & n_8 = 1 \\ \frac{1}{6} a_3 n_9 x + \frac{a_6}{7a_3^2} (a_3 x + a_4)^7 + a_8, & n_8 = 2 \\ -n_9 \left(-\frac{3}{4}\right)^{\frac{3}{4}} (a_1 x + a_5)^{-\frac{3}{4}} \\ -\frac{3a_6}{a_1} \ln(a_1 x + a_5) + a_9, & n_8 \neq 2, n_8 \neq 1, n_8 = -\frac{1}{3} \\ \frac{n_9}{3n_8} (n_8 - 1)^{3-2n_8} (a_1 x + a_5)^{\frac{1}{n_8-1}} \\ + \frac{a_6}{a_1} (n_8 - 1)^{\frac{3+2n_8}{n_8-1}} \frac{1}{1+3n_8} (a_1 x + a_5)^{\frac{1+3n_8}{n_8-1}} + a_{10}, & n_8 \neq 2, \\ & n_8 \neq 1, n_8 \neq -\frac{1}{3} \end{cases} \quad (6.101)$$

where a_6 to a_{10} are arbitrary constants. Using (6.100) and (6.101) in equations (6.93b) and (6.94), we find that the cases where either $n_8 \neq 2, n_8 \neq 1, n_8 = -\frac{1}{3}$ or $n_8 = 1$ or $n_8 \neq 2, n_8 \neq 1, n_8 \neq -\frac{1}{3}$ lead us to a contradiction. Using (6.100) and (6.101) with $n_8 = 2$, in (6.93b), we find that $a_6 = 0$. If $a_6 = 0$, then $f''(x) = 0$ and this case have been studied in section 6.3.3.

$$\text{Subsubcase (3): } \left\{ \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' = 0, \quad \left[\frac{\left(\frac{A_6(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' = 0 \right\}. \quad \text{Using these}$$

two equations in (6.73), we have

$$\left[\frac{\left(\frac{A_4(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' = 0 \quad (6.102)$$

Integrating our two assumptions and equation (6.102) twice with respect to x , we obtain

$$\begin{aligned}A_5(x) &= \lambda_1 A_7(x) + \lambda_2 A_8(x) \\A_6(x) &= \lambda_3 A_7(x) + \lambda_4 A_8(x) \\A_4(x) &= \lambda_5 A_7(x) + \lambda_6 A_8(x)\end{aligned}\tag{6.103}$$

where λ_1 to λ_6 are arbitrary constants of integration. Employing (6.103) in equation (6.66), we get

$$B_3(\psi) = -\lambda_5 B_0(\psi) - \lambda_1 B_1(\psi) - \lambda_3 B_2(\psi)\tag{6.104}$$

Upon substitution of (6.103) and (6.104), equation (6.65) yields

$$B_4(\psi) = -\lambda_6 B_0(\psi) - \lambda_2 B_1(\psi) - \lambda_4 B_2(\psi)\tag{6.105}$$

Integrating equations (6.104) and (6.105) four times with respect to ψ , we have

$$\begin{aligned}[\gamma^3(\psi) + \lambda_3 \gamma^2(\psi) + \lambda_1 \gamma(\psi) + \lambda_5] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= \lambda_7 \gamma^3(\psi) + \lambda_8 \gamma^2(\psi) + \lambda_9 \gamma(\psi) + \lambda_{10} \\[\gamma^4(\psi) + \lambda_4 \gamma^2(\psi) + \lambda_2 \gamma(\psi) + \lambda_6] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= \lambda_{11} \gamma^3(\psi) + \lambda_{12} \gamma^2(\psi) \\&\quad + \lambda_{13} \gamma(\psi) + \lambda_{14}\end{aligned}\tag{6.106a,b}$$

where λ_7 to λ_{14} are arbitrary constants of integration. Dividing equation (6.106b) by (6.106a) and equating coefficients of different powers of $\gamma(\psi)$ to zero, we get

$$\begin{aligned}\lambda_7 &= 0 \\ \lambda_{11} &= \lambda_8 \\ \lambda_{12} &= \lambda_9 - \lambda_3 \lambda_{11} \\ \lambda_{13} &= \lambda_{10} - \lambda_3 \lambda_{12} - \lambda_1 \lambda_8 + \lambda_4 \lambda_8 \\ \lambda_{14} &= \lambda_2 \lambda_8 + \lambda_4 \lambda_9 - \lambda_3 \lambda_{13} - \lambda_1 \lambda_{12} - \lambda_5 \lambda_8 \\ \lambda_3 \lambda_{14} + \lambda_1 \lambda_{13} + \lambda_5 \lambda_{12} - \lambda_4 \lambda_{10} - \lambda_2 \lambda_9 - \lambda_6 \lambda_8 &= 0 \\ \lambda_5 \lambda_{13} - \lambda_2 \lambda_{10} - \lambda_6 \lambda_9 + \lambda_1 \lambda_{14} &= 0 \\ \lambda_5 \lambda_{14} - \lambda_6 \lambda_{10}\end{aligned}\tag{6.107}$$

Using (6.102) to (6.104) in equations (6.64) and (6.63), we get

$$\begin{aligned} A_3(x) + \lambda_8 A_8(x) &= 0 \\ A_2(x) + \lambda_{12} A_8(x) &= 0 \end{aligned} \quad (6.108a,b)$$

Integrating (6.108a) twice with respect to x , we obtain

$$g(x) = \begin{cases} e^{a_1 x + a_2}, & \lambda_8 = 1 \\ a_3 x + a_4, & \lambda_8 = 2 \\ (\lambda_8 - 1)^{\frac{1}{\lambda_8 - 1}} [a_1 x + a_5]^{\frac{1}{\lambda_8 - 1}}, & \lambda_8 \neq 1, \lambda_8 \neq 2 \end{cases}$$

where a_1 to a_5 are arbitrary constants of integration. Using this equation in (6.108b)

and integrating the resulting equation, we have

$$f(x) = \begin{cases} \frac{\lambda_{12}}{3} e^{a_1 x + a_2} + \frac{a_6}{4a_1^3} e^{4(a_1 x + a_2)} + a_7, & \lambda_8 = 1 \\ \frac{1}{6} a_3 \lambda_{12} x + \frac{a_6}{7a_3^3} (a_3 x + a_4)^7 + a_8, & \lambda_8 = 2 \\ -\lambda_{12} \left(-\frac{3}{4}\right)^{\frac{3}{4}} (a_1 x + a_5)^{-\frac{3}{4}} \\ - \frac{3a_6}{a_1} \ln(a_1 x + a_5) + a_9, & \lambda_8 \neq 2, \lambda_8 \neq 1, \lambda_8 = -\frac{1}{3} \\ \frac{\lambda_{12}}{3\lambda_8} (\lambda_8 - 1)^{3-2\lambda_8} (a_1 x + a_5)^{\frac{1}{\lambda_8 - 1}} \\ + \frac{a_6}{a_1} (\lambda_8 - 1)^{\frac{3+2\lambda_8}{\lambda_8 - 1}} \frac{1}{1+3\lambda_8} (a_1 x + a_5)^{\frac{1+3\lambda_8}{\lambda_8 - 1}} + a_{10}, & \lambda_8 \neq 2, \\ & \lambda_8 \neq 1, \lambda_8 \neq -\frac{1}{3} \end{cases}$$

where a_6 to a_{10} are arbitrary constants. Proceeding as in the previous case, we can show that when $\lambda_8 = 2$, then $a_6 = 0$ and this case has been studied in section 6.3.3 and the remaining cases lead us to contradiction.

$$\text{Subsubcase (4): } \left\{ \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' \neq 0, \quad \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' \neq 0 \right\}. \text{ Dividing equation}$$

(6.81) by $\left[\frac{B_2(\psi)}{B_0(\psi)} \right]' \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' \neq 0$, we get

$$\frac{\left[\frac{B_1(\psi)}{B_0(\psi)} \right]'}{\left[\frac{B_2(\psi)}{B_0(\psi)} \right]'} + \frac{\left[\frac{\left(\frac{A_6(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]'}{\left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]'} = 0 \quad (6.109)$$

Equation (6.109) implies that

$$\begin{aligned} \left[\frac{\left(\frac{A_6(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' &= k_1 \left[\frac{\left(\frac{A_5(x)}{A_8(x)} \right)'}{\left(\frac{A_7(x)}{A_8(x)} \right)'} \right]' \\ \left[\frac{B_1(\psi)}{B_0(\psi)} \right]' &= -k_1 \left[\frac{B_2(\psi)}{B_0(\psi)} \right]' \end{aligned} \quad (6.110a,b)$$

where k_1 is an arbitrary constant. Integrating equation (6.110a) twice with respect to x and equation (6.110b) once with respect to ψ , we obtain

$$\begin{aligned} A_6(x) &= k_1 A_5(x) + k_2 A_7(x) + k_3 A_8(x) \\ B_1(\psi) &= -k_1 B_2(\psi) + k_4 B_0(\psi) \end{aligned} \quad (6.111)$$

where k_2 to k_4 are arbitrary constants of integration. Using (6.111), equation (6.75) after two integrations yields

$$A_4(x) = -k_4 A_5(x) + k_5 A_7(x) + k_6 A_8(x) \quad (6.112)$$

where k_5 and k_6 are arbitrary constants. Employing (6.111) and (6.112) in equation (6.66), we get

$$B_3(\psi) = -k_5 B_0(\psi) - k_2 B_2(\psi) \quad (6.113)$$

Substituting (6.111) to (6.113) in equation (6.65), we have

$$B_4(\psi) = -k_6 B_0(\psi) - k_3 B_2(\psi) \quad (6.114)$$

Integrating (6.111b), (6.113) and (6.114) four times with respect to ψ , we obtain

$$\begin{aligned} [k_1 \gamma^2(\psi) + \gamma(\psi) - k_4] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= \frac{k_8}{6} \gamma^3(\psi) + \frac{k_{11}}{2} \gamma^2(\psi) + k_{14} \gamma(\psi) + k_{17} \\ [\gamma^3(\psi) + k_2 \gamma^2(\psi) + k_5] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= \frac{k_9}{6} \gamma^3(\psi) + \frac{k_{12}}{2} \gamma^2(\psi) + k_{15} \gamma(\psi) + k_{18} \\ [\gamma^4(\psi) + k_3 \gamma^2(\psi) + k_6] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} &= \frac{k_7}{6} \gamma^3(\psi) + \frac{k_{10}}{2} \gamma^2(\psi) + k_{13} \gamma(\psi) + k_{16} \end{aligned} \quad (6.115a,b,c)$$

where k_7 to k_{18} are arbitrary constants of integration.

Using (6.111), (6.112) and (6.115) in (6.64), (6.63), (6.62) and (6.61), we get

$$\begin{aligned} A_3(x) &= -\frac{1}{2} k_{12} A_8(x) - \frac{1}{6} k_8 A_5(x) - \frac{1}{6} k_9 A_7(x) \\ A_2(x) &= -\frac{1}{2} [k_{10} A_8(x) + k_{11} A_5(x) + k_{12} A_7(x)] \\ A_1(x) &= -k_{13} A_8(x) - k_{14} A_5(x) - k_{15} A_7(x) \\ A_0(x) &= -k_{16} A_8(x) - k_{17} A_5(x) - k_{18} A_7(x) \end{aligned} \quad (6.116a,b,c,d)$$

Dividing (6.114c) by (6.115a), (6.115b) by (6.115a) and (6.115c) by (6.115b) and equating coefficients of different powers of $\gamma(\psi)$ to zero, we get

$$\begin{aligned} k_8 = k_9 = k_{11} = 0, \quad k_{10} &= -\frac{1}{3} k_2 k_7, \quad k_{12} = \frac{1}{3} k_7, \quad k_{14} = \frac{1}{6} k_1 k_7 \\ k_{17} &= \frac{1}{6} (3 - k_1 k_2) k_7, \quad k_2 k_{17} - k_1 k_{18} - k_{15} + \frac{1}{2} k_4 k_{12}, \quad k_5 k_{17} + k_4 k_{18} = 0 \\ k_5 k_{14} - k_{18} + k_4 k_{15} &= 0, \quad k_{14} k_3 - k_1 k_{13} - \frac{1}{2} k_{10} + \frac{1}{6} k_4 k_7, \\ k_{17} k_3 - k_1 k_{16} - k_{13} + \frac{1}{2} k_4 k_{10} &= 0, \quad k_6 k_{14} - k_{16} + k_4 k_{13} = 0, \\ k_6 k_{17} + k_4 k_{16} - 16 &= 0, \quad k_{18} + \frac{1}{2} k_3 k_{12} - k_{13} - \frac{1}{2} k_2 k_{10} = 0, \\ k_3 k_{15} - \frac{1}{6} k_5 k_7 - k_{16} - k_2 k_{13} &= 0, \quad k_3 k_{13} + \frac{1}{2} k_6 k_{12} - k_2 k_{16} - \frac{1}{2} k_5 k_{10} = 0 \\ k_6 k_{15} - k_5 k_{13} &= 0, \quad k_6 k_{18} - k_5 k_{16} = 0 \\ k_{17} + \frac{1}{6} k_1 k_2 k_7 - k_1 k_{15} - \frac{1}{6} k_7 &= 0, \quad k_{17} - \frac{1}{6} k_7 + \frac{1}{6} k_1 k_2 k_7 = 0 \end{aligned} \quad (6.117)$$

The last two equations of system (6.117) give

$$k_1 k_{15} = 0 \quad (6.118)$$

Equation (6.118) implies that either $k_1 = 0$ or $k_{15} = 0$. If $k_1 = 0$, then system (6.117) implies one of the following four possibilities

- (a) $k_1 = k_8 = k_9 = k_{11} = k_{14} = k_7 = k_{12} = k_{17} = 0$
- (b) $k_1 = k_8 = k_9 = k_{11} = k_{14} = k_{15} = k_{18} = k_{17} = k_{12} = 0, \quad k_2 + k_4 = 0$
- (c) $k_1 = k_8 = k_9 = k_{11} = k_{14} = k_{15} = k_{18} = k_5 = k_3 = 0, \quad k_2 + k_4 = 0$
- (d) $k_1 = k_8 = k_9 = k_{11} = k_{14} = k_{15} = k_{18} = k_5 = k_{13} = k_{16} = k_6 = 0,$
 $k_3 + k_2^2 = 0, \quad k_2 + k_4 = 0$

When $k_{15} = 0$ is taken in system (6.117), then we have the following three possibilities

- (e) $k_{15} = k_8 = k_9 = k_{11} = k_5 = k_{18} = k_3 = 0$
- (f) $k_{15} = k_8 = k_9 = k_{11} = k_{13} = k_3 = 0$
- (g) $k_{15} = k_8 = k_9 = k_{11} = k_5 = k_{18} = 0, \quad k_3 + k_2^2 = 0$

For possibility (a), since $k_8 = k_{11} = k_{14} = k_{17} = 0$, then equation (6.115a) implies $\gamma''(\psi) = 0$ which contradicts our assumption that $\gamma''(\psi) \neq 0$. For possibility (b), since $k_9 = k_{12} = k_{15} = k_{18} = 0$, then equation (6.115b) implies that $\gamma''(\psi) = 0$ which contradicts our assumption. Possibilities (c) to (g) need further consideration.

Integrating equation (6.116a) with $k_8 = k_9 = 0$ twice with respect to x , we obtain

$$g(x) = \begin{cases} a_1 x + a_2, & k_7 = 12 \\ e^{a_3 x + a_4}, & k_7 = 6 \\ \left(\frac{k_7 - 6}{6} \right)^{\frac{a}{k_7 - 6}} (a_3 x + a_5)^{\frac{a}{k_7 - 6}}, & k_7 \neq 12, k_7 \neq 6 \end{cases} \quad (6.119)$$

where a_1 to a_5 are arbitrary constants. Employing $k_{11} = 0$, $k_{10} = -\frac{1}{3}k_2 k_7$, $k_{12} =$

$\frac{1}{3}k_7$ and (6.119) in (6.116b) and integrating the resulting equation, we have

$$f(x) = \begin{cases} 2k_2a_1x - \frac{a_6}{a_1^3} \frac{1}{a_1x + a_2} + a_7, & k_7 = 12 \\ k_2e^{a_3x+a_4} + \frac{a_6}{a_3^2}x + a_8, & k_7 = 6 \\ \frac{k_2}{2} \left(\frac{k_7-6}{6} \right)^{\frac{6}{k_7-6}} (a_3x + a_5)^{\frac{6}{k_7-6}} \\ - \frac{a_6}{a_3^3} \left(\frac{6}{k_7-6} \right)^2 \frac{1}{a_3x + a_5} + a_9, & k_7 \neq 12, k_7 \neq 6 \end{cases} \quad (6.120)$$

where a_6 to a_9 are arbitrary constants.

Possibility (c): $\{k_1 = k_8 = k_9 = k_{11} = k_{14} = k_{15} = k_{18} = k_5 = k_3 = 0, k_2 + k_4 = 0\}$

Substituting (6.119) and (6.120) with $k_7 = 12$ in equation (6.111a) with $k_1 = k_3 = 0$, we find that $2 + 16k_2^2a_1^2 = 0$ which is not possible. Similarly, the other two cases lead us to a contradiction.

Possibility (d): $\{k_1 = k_8 = k_9 = k_{11} = k_{14} = k_{15} = k_{18} = k_5 = k_{13} = k_{16} = k_6 = 0, k_3 + k_2^2 = 0, k_2 + k_4 = 0\}$. Equation (6.111a) with $k_1 = 0$ and $k_3 = -k_2^2$ gives

$$6f'^2(x) - 4k_2g'(x)f'(x) + k_2^2g'^2(x) = 0$$

Solving this equation for $f'(x)$, we obtain

$$f'(x) = \frac{4k_2g'(x) \pm 2i\sqrt{12 + 2k_2^2g'^2(x)}}{12}$$

This equation implies that $f(x)$ is an imaginary function which is not possible.

Possibility (e): $\{k_{15} = k_8 = k_9 = k_{11} = k_5 = k_{18} = k_3 = 0\}$. For this case $f(x)$ and $g(x)$ are given by equations (6.119) and (6.120) respectively. Following the procedure above, we can show that when $k_7 = 6$ and $k_7 \neq 6, k_7 \neq 12$ are not possible. When $k_7 = 12$, then we find that $a_6 = 0$ and this case was studied in section 6.3.3.

Possibility (f): $\{k_{15} = k_8 = k_9 = k_{11} = k_{13} = k_3 = 0\}$ Proceeding as in possibility (e), we can show that the case where $k_7 = 12$ implies that $a_6 = 0$ and this

corresponds to the case studied in section 6.3.3. The other two cases are not possible.

Possibility (g): $\{k_{15} = k_8 = k_9 = k_{11} = k_5 = k_{13} = 0, \quad k_3 + k_2^2 = 0\}$. The conclusions of possibility (e) are also valid for this possibility.

Summing up all the results of this section, we have the following theorem:

Theorem 6.5. *There does not exist any rotational flow for steady plane infinitely conducting MHD orthogonal flow with streamline pattern of the form $\frac{y - f(x)}{g(x)} = \text{constant}$ and the only possible irrotational flows are the ones with parallel straight lines or concurrent straight lines.*

6.5 SOLUTIONS FOR MHD VARIABLELY-INCLINED FLOW.

In this section, we study two examples for infinitely conducting MHD variably-inclined flows to show the use of this method for general steady plane MHD flows. The reason for studying only these two examples is the complexity of the equations.

Letting $\frac{y - f(x)}{g(x)} = \text{constant}$ be the streamlines, we get

$$\frac{y - f(x)}{g(x)} = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (6.121)$$

where $\gamma'(\psi)$ is the derivative of the unknown function $\gamma(\psi)$.

Employing (6.121) and $\phi = x$ in (2.24) and (2.26), we have

$$\begin{aligned} E &= 1 + [f'(x) + g'(x)\gamma(\psi)]^2, & G &= g^2(x)\gamma'^2(\psi) \\ F &= [f'(x) + g'(x)\gamma(\psi)]g(x)\gamma'(\psi), & J = W &= g(x)\gamma'(\psi) \end{aligned} \quad (6.122)$$

Using (6.122) and $\phi = x$, equations (2.34) gives

$$\begin{aligned} \Gamma_{11}^2 &= \frac{1}{g(x)\gamma'(\psi)} [f''(x) + g''(x)\gamma(\psi)] \\ \Gamma_{12}^2 &= \frac{g'(x)}{g(x)} \end{aligned} \quad (6.123)$$

6.5.1 Example I. (Parabolic flows along $y - m_1x^2 - m_2x = \text{constant}$).

For these flows, we have

$$y - m_1x^2 - m_2x = \gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (6.124)$$

Comparing (6.124) with (6.121), we obtain

$$f(x) = m_1x^2 + m_2x, \quad g(x) = 1 \quad (6.125)$$

Using (6.122) and (6.125) in equation (6.20), we get

$$H = \frac{C}{\sqrt{1 + [2m_1x + m_2]^2}} \gamma'(\psi) \csc \theta \quad (6.126)$$

Employing (6.122), (6.123), (6.125) and (6.126) in equation (6.18), we obtain

$$\frac{2m_1}{[1 + (2m_1x + m_2)^2]^2} [(2m_1x + m_2)^2 - 1] - \frac{4m_1(2m_1x + m_2)}{[1 + (2m_1x + m_2)^2]^2} \cot\theta - \frac{1}{1 + (2m_1x + m_2)^2} \frac{\partial\theta}{\partial x} \csc^2\theta + \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0$$

which upon integration with respect to x gives

$$\cot\theta = (2m_1x + m_2) + \left[-\frac{\gamma''(\psi)}{\gamma'^2(\psi)}x + A(\psi) \right] [1 + (2m_1x + m_2)^2] \quad (6.127)$$

where $A(\psi)$ is an arbitrary function of ψ .

Substituting (6.122), (6.123), (6.125), (6.126) and (6.127) in equation (6.19), we get after some calculations

$$\begin{aligned} \Omega = \frac{C}{\gamma'(\psi)\sqrt{1 + (2m_1x + m_2)^2}} & \left\{ 2m_1A(\psi)\gamma'^2(\psi)\sqrt{1 + (2m_1x + m_2)^2} \right. \\ & - (4m_1x + 2m_2)\sqrt{1 + (2m_1x + m_2)^2}\gamma''(\psi) - [1 + (2m_1x + m_2)^2]^{\frac{3}{2}} [A(\psi)\gamma'(\psi)]' \\ & \left. + x[1 + (2m_1x + m_2)^2] \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \right\} \end{aligned} \quad (6.128)$$

Employing (6.15), (6.122), (6.125) to (6.128) in equation (6.17), we obtain after some simplifications

$$\sum_{n=0}^4 B_n(\psi) x^n = 0 \quad (6.129)$$

where the coefficients $B_n(\psi)$ are given by

$$\begin{aligned} B_4(\psi) &= 16m_1^4 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\ B_3(\psi) &= 32m_1^3m_2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{\mu^*}{\mu} C^2 \left\{ 4m_1^2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \right. \\ & \quad \left. - 16m_1^2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \frac{\gamma''(\psi)}{\gamma'(\psi)} \right\} \end{aligned}$$

$$\begin{aligned}
B_2(\psi) &= -48m_1^3 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + (24m_1^2 m_2^2 + 8m_1^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad + \frac{\mu^*}{\mu} C^2 \left\{ 4m_1 m_2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] - 4m_1^2 [A(\psi)\gamma'(\psi)]'' \right. \\
&\quad + 4m_1^2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] A(\psi)\gamma'(\psi) + 12m_1^2 [A(\psi)\gamma'(\psi)]' \frac{\gamma''(\psi)}{\gamma'(\psi)} \\
&\quad \left. - 12m_1 m_2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \frac{\gamma''(\psi)}{\gamma'(\psi)} \right\} \\
B_1(\psi) &= -48m_1^2 m_2 \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + 8m_1 m_2 (1 + m_2^2) \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' - \frac{8m_1^2 \rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \\
&\quad + \frac{\mu^*}{\mu} C^2 \left\{ -6m_1 \gamma'''(\psi) + (1 + m_2^2) \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right]' \right. \\
&\quad + 8m_1 m_2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] A(\psi)\gamma'(\psi) - 4m_1 m_2 [A(\psi)\gamma'(\psi)]'' \\
&\quad + 12m_1 \frac{\gamma''^2(\psi)}{\gamma'(\psi)} + 8m_1 m_2 [A(\psi)\gamma'(\psi)]' \frac{\gamma''(\psi)}{\gamma'(\psi)} - 8m_1^2 A(\psi) [A(\psi)\gamma'(\psi)]' \gamma'(\psi) \\
&\quad \left. - 2(1 + m_2^2) \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \frac{\gamma''(\psi)}{\gamma'(\psi)} \right\} \\
B_0(\psi) &= 12m_1^2 \frac{\gamma''(\psi)}{\gamma'^2(\psi)} - 2m_1 (2 + 6m_2^2) \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' + (1 + m_2^2)^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \\
&\quad - \frac{4m_1 m_2 \rho}{\mu} \frac{\gamma''(\psi)}{\gamma'^3(\psi)} + \frac{\mu^*}{\mu} C^2 \left\{ -4m_1 A(\psi)\gamma'(\psi)\gamma''(\psi) + 2m_1 \gamma'^2(\psi) A'(\psi) \right. \\
&\quad - 2m_2 \gamma'''(\psi) + (1 + m_2^2) \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] A(\psi)\gamma'(\psi) \\
&\quad - (1 + m_2^2) [A(\psi)\gamma'(\psi)]'' + 2m_2 \frac{\gamma''^2(\psi)}{\gamma'(\psi)} + (1 + m_2^2) [A(\psi)\gamma'(\psi)]' \frac{\gamma''(\psi)}{\gamma'(\psi)} \\
&\quad \left. - 4m_1 m_2 A(\psi) [A(\psi)\gamma'(\psi)]' \gamma'(\psi) \right\}
\end{aligned}$$

Equation (6.129) is a fourth degree polynomial in x with coefficients as functions of ψ only. Since this equation must hold true for all values of x , we get

$$B_4(\psi) = B_3(\psi) = B_2(\psi) = B_1(\psi) = B_0(\psi) = 0 \quad (6.130)$$

Requiring $B_4(\psi) = 0$, we have the following two cases:

$$\begin{aligned} \text{a) } m_1 \neq 0, \quad & \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \\ \text{b) } m_1 = 0, \quad & \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0 \end{aligned}$$

We shall study these two cases separately in the following:

Case (a): $\left\{ m_1 \neq 0, \quad \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0 \right\}.$

Integrating $\left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' = 0$ twice with respect to ψ , we obtain

$$\frac{\gamma''(\psi)}{\gamma'^3(\psi)} = a_1 \gamma(\psi) + a_2 \quad (6.131)$$

where a_1 and a_2 are arbitrary constants. Upon substitution of (6.131), equation $B_3(\psi) = 0$ yields

$$a_1 = 0 \quad (6.132)$$

Employing (6.131) and (6.132) in equation $B_2(\psi) = 0$, we get

$$m_1 [A(\psi)\gamma'(\psi)]'' - 3m_1 a_2 \{A(\psi)\gamma'^3(\psi)\}' - m_2 a_2^2 (\gamma'^2(\psi))' \gamma'^2(\psi) = 0 \quad (6.133)$$

which after one integration gives

$$m_1 A'(\psi)\gamma'(\psi) - 2m_1 a_2 A(\psi)\gamma'^3(\psi) - \frac{1}{2} m_2 a_2^2 \gamma'^4(\psi) = \lambda_1 \quad (6.134)$$

where λ_1 is an arbitrary constant of integration. Using (6.131) with $a_1 = 0$ and (6.134) in $B_2(\psi) = 0$, we have

$$\begin{aligned} & -\frac{8m_1^2 \rho}{\mu} + \frac{\mu^*}{\mu} C^2 \{ -6m_1 a_2^2 \gamma'^5(\psi) + 4(1+m_2^2) a_2^3 \gamma'^6(\psi) - 8m_1 m_2 a_2^2 A(\psi)\gamma'^5(\psi) \\ & - 12m_1 m_2 a_2 [A(\psi)\gamma'(\psi)]' - 8m_2^2 a_2^3 \gamma'^6(\psi) + 8m_1 m_2 a_2 [A(\psi)\gamma'(\psi)]' \gamma'^2(\psi) \\ & - 12m_1^2 a_2 A^2(\psi)\gamma'^3(\psi) - 8m_1 m_2 a_2^2 \gamma'^4(\psi) - 8m_1 \lambda_1 A(\psi)\gamma'(\psi) \} = 0 \end{aligned} \quad (6.135)$$

Differentiating equation (6.135) twice with respect to ψ and using equation (6.134), we obtain

$$\begin{aligned}
a_2 \left\{ -6m_1 a_2 [\gamma'^5(\psi)]'' + 4(1 + m_2^2) a_2^2 [\gamma'^6(\psi)]'' - 12m_1 m_2 [A(\psi) \gamma'(\psi)]''' \right. \\
- 8m_1 m_2 a_2 [A(\psi) \gamma'^5(\psi)]'' - 8m_2^2 a_2^2 [\gamma'^6(\psi)]'' - 8m_1 m_2 a_2 [\gamma'^4(\psi)]'' \\
+ 8m_1 m_2 \left([A(\psi) \gamma'(\psi)]' \gamma'^2(\psi) \right)'' - 12m_1^2 [A^2(\psi) \gamma'^3(\psi)]'' \\
\left. - 24\lambda_1 m_1 [A(\psi) \gamma'^3(\psi)]' - 8\lambda_1 m_2 a_2 \gamma'^2(\psi) [\gamma'^2(\psi)]' \right\} = 0
\end{aligned} \tag{6.136}$$

Two cases arise from equation (6.136): (i) $a_2 = 0$ and (ii) $a_2 \neq 0$.

Subcase (i): $\{a_2 = 0\}$. Using $a_1 = a_2 = 0$ in equation (6.131), we get

$$\gamma''(\psi) = 0$$

which upon integration gives

$$\gamma(\psi) = a_3 \psi + a_4 \tag{6.137}$$

where $a_3 \neq 0$ and a_4 are arbitrary constants. Employing $a_2 = 0$ and (6.137) in equation (6.133) and integrating the resulting equation we have

$$A(\psi) = a_5 \psi + a_6 \tag{6.138}$$

where a_5 and a_6 are arbitrary constants. Substituting (6.137) and (6.138) in $B_0(\psi) = 0$, we find that

$$a_5 = 0 \tag{6.139}$$

Thus, the family of curves $y - m_1 x^2 - m_2 x = \text{constant}$ are permissible streamlines for infinitely conducting MHD variably-inclined flow with $\gamma(\psi)$ given by equation

(6.137). The exact solutions of this flow pattern are given by

$$\begin{aligned}
u &= \frac{1}{a_3}, & v &= \frac{1}{a_3} (2m_1x + m_2), \\
\cot\theta &= (2m_1x + m_2) + a_3 \left[1 + (2m_1x + m_2)^2 \right] \\
H &= \frac{Ca_3}{\sqrt{1 + (2m_1x + m_2)^2}} \csc\theta \\
\omega &= \frac{2m_1}{a_3}, & \Omega &= 2m_1a_3a_3C \\
p &= -2\mu^*m_1a_3^2C^2x + \left(2\mu^*m_1a_3a_3^2C^2 - \frac{2m_1\rho}{a_3^2} \right) [y - m_1x^2 - m_2x - a_4] \\
&\quad - \frac{\rho}{2a_3^2} \left[1 + (2m_1x + m_2)^2 \right] + p_0
\end{aligned} \tag{6.140}$$

where p_0 is an arbitrary constant.

Subcase (ii): $\{a_2 \neq 0\}$. Employing equations (6.131) to (6.135) in $B_0(\psi) = 0$, we obtain

$$D_1(\psi)A(\psi) + D_2(\psi) = 0 \tag{6.141}$$

where $D_1(\psi)$ and $D_2(\psi)$ are given by

$$\begin{aligned}
D_1(\psi) &= \frac{\mu^*}{\mu} C^2 \{ 4m_2\lambda_1\gamma'(\psi) - 9a_2^2(1 - 3m_2^2)\gamma'^3(\psi) - a_2^2(1 + 21m_2^2)\gamma'^5(\psi) \} \\
D_2(\psi) &= -12m_1^2a_2\gamma'(\psi) + \frac{4m_1m_2a_2\rho}{\mu} + \frac{\mu^*}{\mu} C^2 \{ -6m_2a_2^2\gamma'^5(\psi) + 2\lambda_1\gamma'(\psi) \\
&\quad - \frac{3m_2}{m_1}(1 - 3m_2^2)a_2^3\gamma'^4(\psi) - \frac{3a_2\lambda_1}{m_1}(1 - 3m_2^2) - \frac{5m_2}{m_1}(1 + m_2^2)a_2^3\gamma'^6(\psi) \\
&\quad + \frac{\lambda_1a_2}{m_1}(1 + m_2^2)\gamma'^2(\psi) + 8m_2^2a_2^2\gamma'^4(\psi) \}
\end{aligned}$$

If $D_1(\psi) = 0$, then we find that $a_2 = 0$ which contradicts our assumption that $a_2 \neq 0$. Therefore, $D_1(\psi) \neq 0$ and equation (6.141) gives

$$A(\psi) = \frac{D_2(\psi)}{D_1(\psi)} \tag{6.142}$$

Using (6.131) and (6.142), equation (6.134) after some calculations yields a polynomial of degree twelve in $\gamma(\psi)$ with constant coefficients. Since $\gamma(\psi) \neq \text{constant}$, then

the coefficients of different power of $\gamma(\psi)$ must be zero. Requiring the coefficient of $\gamma^{12}(\psi)$ to be zero, we get

$$16m_2^2 \frac{\lambda_1}{m_1} a_2^{12} = 0$$

This equation implies that either $m_2 = 0$ or $\lambda_1 = 0$. If $m_2 = 0$ is taken in the coefficient of $\gamma^6(\psi)$, then we get $m_1 a_2 = 0$ which contradicts our assumptions that $m_1 \neq 0$ and $a_2 \neq 0$. If $\lambda_1 = 0$ is taken in the coefficient of $\gamma^9(\psi)$, we find that $m_2 = \pm \frac{1}{\sqrt{3}}$. Using $m_2 = \pm \frac{1}{\sqrt{3}}$ and $\lambda_1 = 0$ in the coefficient of $\gamma^6(\psi)$, we get $m_1 a_2 = 0$ which contradicts our assumptions. Thus, the case where $a_2 \neq 0$ is not possible.

Case (b): $\left\{ m_1 = 0, \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' \neq 0 \right\}.$

Using $m_1 = 0$ in $B_3(\psi) = B_2(\psi) = B_1(\psi) = B_0(\psi) = 0$, we find that $B_3(\psi) = 0$, $B_2(\psi) = 0$ are identically satisfied and $B_1(\psi) = 0$, $B_0(\psi) = 0$ gives

$$\begin{aligned} & \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] - 2 \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] \frac{\gamma''(\psi)}{\gamma'(\psi)} = 0 \\ & (1 + m_2^2)^2 \left[\frac{1}{\gamma'(\psi)} \left(\frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right)' \right]' + \frac{\mu^*}{\mu} C^2 \left\{ -2m_2 \gamma'''(\psi) + 2m_2 \frac{\gamma''^2(\psi)}{\gamma'(\psi)} \right. \\ & \quad + (1 + m_2^2) \left[\gamma'(\psi) \left(\frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right)' + \frac{\gamma''^2(\psi)}{\gamma'^2(\psi)} \right] A(\psi) \gamma'(\psi) \\ & \quad \left. - (1 + m_2^2) [A(\psi) \gamma'(\psi)]'' + (1 + m_2^2) [A(\psi) \gamma'(\psi)]' \frac{\gamma''(\psi)}{\gamma'(\psi)} \right\} = 0 \end{aligned} \quad (6.143a,b)$$

A solution of equation (6.143a) is given by

$$\gamma(\psi) = \frac{1}{a_7} e^{a_7 \psi + a_8} + a_9 \quad (6.144)$$

where $a_7 \neq 0$, a_8 and a_9 are arbitrary constants. Using (6.144) in (6.143b) and integrating the resulting equation, we obtain

$$A(\psi) = \frac{\mu}{\mu^* C^2} \left\{ (1 + m_2^2) a_7 e^{-2(a_7 \psi + a_8)} + \frac{a_{10}}{a_7} - a_{11} \right\} \quad (6.145)$$

where a_{10} and a_{11} are arbitrary constants of integration.

Thus, $y - m_2x = \text{constant}$ are possible streamlines for infinitely conducting MHD variably-inclined flow with $\gamma(\psi)$ given by equation (6.144). The exact solutions of this rotational flow are given by

$$\begin{aligned}
u &= \frac{1}{a_7} \frac{1}{y - m_2x - a_9}, & v &= \frac{m_2}{a_7} \frac{1}{y - m_2x - a_9} \\
\cot \theta &= m_2 - (1 + m_2^2) \frac{x}{y - m_2x - a_9} \\
&\quad + \frac{\mu(1 + m_2^2)}{\mu^* C^2 a_7} \left[\frac{1 + m_2^2}{(y - m_2x - a_9)^2} + a_9 - a_7 a_{10} \right] \\
H &= \frac{C a_7}{\sqrt{1 + m_2^2}} (y - m_2x - a_9) \csc \theta \\
\omega &= \frac{1 + m_2^2}{a_7 (y - m_2x - a_9)^2} \\
\Omega &= -2m_2 a_7 C + \frac{\mu}{\mu^* C} \frac{(1 + m_2^2)^2}{(y - m_2x - a_9)^2} - \frac{\mu a_7}{\mu^* C} (1 + m_2^2) (a_9 - a_7 a_{10}) \\
p &= \frac{\mu(1 + m_2^2)(2a_7 - 1 - m_2^2)x}{y - m_2x - a_9} + \mu(1 + m_2^2)(a_9 - a_7 a_{10}) a_7 (y - m_2x - a_9) x \\
&\quad + 2\mu^* a_7 m_2 C^2 x - \frac{1}{2} (1 + m_2^2) \left[2\mu m_2 + \frac{\mu^2}{\mu^* C^2} (1 + m_2^2)^2 \right] \frac{1}{a_7 (y - m_2x - a_9)^2} \\
&\quad - \frac{1}{2} (a_9 - a_7 a_{10}) \left[\frac{\mu^2}{\mu^* C^2} (1 + m_2^2) (a_9 - a_7 a_{10}) + \mu m_2 a_7 \right] (y - m_2x - a_9)^2 \\
&\quad - 2\mu m_2 (1 + m_2^2) \ln [a_7 (y - m_2x - a_9)] - \frac{\mu^2}{\mu^* C^2} (1 + m_2^2) (a_9 - a_7 a_{10}) + p_0
\end{aligned} \tag{6.146}$$

where p_0 is an arbitrary constant.

Summing up the above results, we have:

Theorem 6.6. *Streamline pattern $y - m_1x^2 - m_2x = \text{constant}$ of steady plane motion is permissible for infinitely conducting MHD variably-inclined flow with solutions given by equations (6.140) if $m_1 \neq 0$ and a solution given by equations (6.146) if $m_1 = 0$.*

6.5.2 Example II. (Flow along $\frac{y}{x} = \text{constant}$ as streamlines).

We assume that

$$y = x\gamma(\psi); \quad \gamma'(\psi) \neq 0 \quad (6.147)$$

Comparing (6.147) with (6.121), we have

$$f(x) = 0, \quad g(x) = x \quad (6.148)$$

Using (6.122) and (6.148) in equation (6.20), we get

$$H = \frac{Cx}{\sqrt{1 + \gamma^2(\psi)}} \gamma'(\psi) \csc \theta \quad (6.149)$$

Employing (6.122), (6.123), (6.148) and (6.149) in equation (6.18), we obtain

$$-x \frac{\partial \theta}{\partial x} \csc^2 \theta + 2 \cot \theta - 2\gamma(\psi) + [1 + \gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} = 0$$

which upon integration with respect to x yields

$$\cot \theta = \frac{1}{2} \left\{ 2\gamma(\psi) - [1 + \gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right\} + \frac{D(\psi)}{x^2} \quad (6.150)$$

where $D(\psi)$ is an arbitrary function of ψ .

Upon substitution of (6.122), (6.123), (6.148) to (6.150), equation (6.19) after some calculations gives

$$\begin{aligned} \Omega = C \left\{ \gamma'(\psi) - \frac{\gamma(\psi)\gamma''(\psi)}{\gamma'(\psi)} - \frac{1}{2} [1 + \gamma^2(\psi)] \left[\frac{\gamma''^2(\psi)}{\gamma'^3(\psi)} - \frac{\gamma'''(\psi)}{\gamma'^2(\psi)} \right] \right\} \\ - \frac{1}{x^2} \frac{1}{\gamma'(\psi)} [D(\psi)\gamma'(\psi)]' \end{aligned} \quad (6.151)$$

Using (6.15), (6.122), (6.148) to (6.151) in equation (6.17), after some simplifications we obtain

$$\sum_{n=0}^2 A_{2n}(\psi) x^{2n} = 0 \quad (6.152)$$

where $A_n(\psi)$ are given by

$$\begin{aligned}
A_4(\psi) &= \left\{ \frac{\gamma'(\psi)}{\sqrt{1+\gamma^2(\psi)}} - \frac{\gamma(\psi)\gamma''(\psi)}{\sqrt{1+\gamma^2(\psi)}\gamma'(\psi)} - \frac{1}{2}\sqrt{1+\gamma^2(\psi)} \left[\frac{\gamma''^2(\psi)}{\gamma'^3(\psi)} - \frac{\gamma''^3(\psi)}{\gamma'^2(\psi)} \right] \right\}' \\
&\quad \left\{ \frac{\gamma'(\psi)}{\sqrt{1+\gamma^2(\psi)}} - \frac{\gamma(\psi)\gamma''(\psi)}{\sqrt{1+\gamma^2(\psi)}\gamma'(\psi)} - \frac{1}{2}\sqrt{1+\gamma^2(\psi)} \left[\frac{\gamma''^2(\psi)}{\gamma'^3(\psi)} - \frac{\gamma''^3(\psi)}{\gamma'^2(\psi)} \right] \right\} \frac{\gamma''(\psi)}{\gamma'(\psi)} \\
A_2(\psi) &= [D(\psi)\gamma'(\psi)]'' \\
A_0(\psi) &= 6\gamma'(\psi) \left\{ -2\frac{\gamma(\psi)}{\gamma'(\psi)} + [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\} \\
&\quad + 4\gamma(\psi) \left\{ -2\frac{\gamma(\psi)}{\gamma'(\psi)} + [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}' + \frac{2\mu^*C^2}{\mu} D(\psi)\gamma'(\psi) \frac{[D(\psi)\gamma'(\psi)]'}{1+\gamma^2(\psi)} \\
&\quad + [1+\gamma^2(\psi)] \frac{1}{\gamma'(\psi)} \left\{ -2\frac{\gamma(\psi)}{\gamma'(\psi)} + [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}'' \\
&\quad + \left\{ 2\gamma(\psi) - [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^2(\psi)} \right\} \left\{ -2\frac{\gamma(\psi)}{\gamma'(\psi)} + [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}' \\
&\quad + \frac{2\rho}{\mu} \left\{ -2\frac{\gamma(\psi)}{\gamma'(\psi)} + [1+\gamma^2(\psi)] \frac{\gamma''(\psi)}{\gamma'^3(\psi)} \right\}
\end{aligned}$$

Equation (6.152) is a fourth degree polynomial in x with coefficients as functions of ψ only. Since x, ψ are independent variables and equation (6.152) must hold true for all values of x , we have

$$A_0(\psi) = A_2(\psi) = A_4(\psi) = 0 \quad (6.153)$$

Requiring $A_4(\psi) = 0$, we get

$$\gamma^{(iv)}(\psi) - 4\frac{\gamma''(\psi)\gamma'''(\psi)}{\gamma'(\psi)} + 3\frac{\gamma''^3(\psi)}{\gamma'^2(\psi)} = 0$$

which upon integration three times with respect to ψ gives

$$\gamma'(\psi) = \frac{b_1}{2}\gamma^2(\psi) + b_2\gamma(\psi) + b_3 \quad (6.154)$$

where b_1 to b_3 are arbitrary constants that are not zero simultaneously.

Integrating $A_2(\psi) = 0$ twice with respect to ψ , we get

$$D(\psi) = \frac{c_1\psi + c_2}{\gamma'(\psi)} \quad (6.155)$$

Using (6.154) and (6.155) in $B_0(\psi) = 0$, after some calculations we get

$$\sum_{n=0}^6 [M_n + (c_1\psi + c_2) N_n] \gamma^n(\psi) = 0 \quad (6.156)$$

where

$$\begin{aligned} M_0 &= -3b_1b_2b_3 + \frac{\rho}{\mu}b_2b_3 + 3b_2^3 + 6b_2b_3^2 \\ M_1 &= -12b_3^3 + 6b_1b_2b_3 + 6b_1b_2^2 - 3b_1^2b_3 + 12b_1b_3^2 + \frac{\rho}{\mu}(b_2^2 + b_1b_3 - 2b_3^2) \\ M_2 &= -12b_2b_3^2 + 3b_2^3 - 9b_1b_2b_3 + \frac{\rho}{2\mu}(9b_1^2b_2 + 3b_1b_2 - 4b_2b_3) \\ M_3 &= 18b_1b_3^2 - 12b_3^3 + 6b_1b_2^2 - 9b_1^2b_3 - 12b_2^2b_3 + \frac{3}{2}b_1^3 + \frac{\rho}{2\mu}(b_1^2 - 4b_3^2) \\ M_4 &= 3b_1b_2b_3 - 3b_2^3 + 3b_1^2b_2 - 18b_2b_3^2 + \frac{\rho}{\mu}(b_1b_2 - 3b_2b_3) \\ M_5 &= -6b_1^2b_3 + 6b_1b_3^2 - 12b_2^2b_3 + \frac{3}{2}b_1^3 + \frac{\rho}{2\mu}(b_1^2 - 2b_1b_3 - b_2^2) \\ M_6 &= -\frac{3}{2}b_1^2b_2 + 6b_1b_2b_3 - 3b_2^3 - \frac{\rho}{2\mu}b_1b_2 \\ N_0 &= \frac{\mu^*C^2}{\mu}c_1b_3^3 \\ N_1 &= \frac{3\mu^*C^2}{\mu}c_1b_2b_3 \\ N_2 &= \frac{\mu^*C^2}{\mu}c_1\left(\frac{3}{2}b_1b_3^2 + 3b_2^2b_3\right) \\ N_3 &= \frac{\mu^*C^2}{\mu}c_1(3b_1b_2b_3 + b_2^2) \\ N_4 &= \frac{\mu^*C^2}{\mu}c_1\left(\frac{3}{4}b_1^2b_3 + \frac{3}{2}b_1b_2^2\right) \\ N_5 &= \frac{3\mu^*C^2}{4\mu}c_1b_1^2 \\ N_6 &= \frac{\mu^*C^2}{8\mu}c_1b_1^3 \end{aligned}$$

Differentiating equation (6.156) six times with respect to ψ , we get

$$\begin{aligned} &720M_6 - 80\frac{\mu^*C^2}{\mu}c_1^2b_1^2[b_1\gamma(\psi) + b_2]\frac{1}{\gamma'(\psi)} - 80\frac{\mu^*C^2}{\mu}c_1b_1^3(c_1\psi + c_2) \\ &- 108\frac{\mu^*C^2}{\mu}c_1^2b_1^2[b_1\gamma(\psi) + b_2] + 24\frac{\mu^*C^2}{\mu}c_1^2b_1[h_1\gamma(\psi) + \dot{u}_2]^3 \\ &- 6\frac{\mu^*C^2}{\mu}c_1^2[b_1\gamma(\psi) + b_2]^5\frac{1}{\gamma'(\psi)} = 0 \end{aligned} \quad (6.157)$$

where $\gamma'(\psi)$ is given by equation (6.154). Differentiating equation (6.157) once again with respect to ψ , we have

$$\sum_{n=0}^6 L_n \gamma^n(\psi) = 0 \quad (6.158)$$

with $L_6 = -36c_1^2 b_1^2$. Requiring equation (6.158) to hold true for all values of $\gamma(\psi)$, the coefficients of different powers of $\gamma(\psi)$ must vanish simultaneously. Taking $L_6 = 0$, we get

$$c_1 b_1 = 0$$

which implies the following cases: (1) $b_1 = 0$ or (2) $c_1 = 0$.

Case (1): $\{b_1 = 0\}$.

Substituting $b_1 = 0$ in equation (6.157), we get

$$-2160b_2^3 [b_2\gamma(\psi) + b_3] - 6\frac{\mu^* C^2}{\mu} c_1^2 b_2^5 = 0 \quad (6.159)$$

Equation (6.159) implies that

$$b_2 = 0$$

since $\gamma(\psi) \neq \text{constant}$. Employing $b_1 = b_2 = 0$ in equation (6.156), we obtain

$$\begin{aligned} c_1 &= 0 \\ b_3 &= -\frac{\rho}{6\mu} \end{aligned} \quad (6.160)$$

Using $b_1 = b_2 = 0$ and $b_3 = -\frac{\rho}{6\mu}$ in equation (6.154) and integrating the resulting equation, we have

$$\gamma(\psi) = -\frac{\rho}{6\mu}\psi + b_4 \quad (6.161)$$

where b_4 is an arbitrary constant.

Thus, $\frac{y}{x} = \text{constant}$ are permissible streamlines for infinitely conducting MHD variably-inclined flow with $\gamma(\psi)$ given by equation (6.161). The exact solutions for

this rotational flow are given by

$$\begin{aligned}
u &= -\frac{6\mu}{\rho x}, & v &= -\frac{6\mu y}{\rho x^2} \\
\cot \theta &= \frac{y}{x} - \frac{6\mu c_2}{\rho x^2} \\
H &= -\frac{\rho C}{6\mu} \frac{x^2}{\sqrt{x^2 + y^2}} \csc \theta \\
\omega &= \frac{12\mu y}{\rho x^3}, & \Omega &= -\frac{\rho C}{6\mu} \\
p &= -\frac{12\mu}{\rho x^2} - \frac{\mu^* \rho^2 C^2}{72\mu^2} x^2 + \mu^* c_2 C^2 \tan^{-1} \left(\frac{y}{x} \right) + p_0
\end{aligned} \tag{6.162}$$

where p_0 is an arbitrary constant. Since the pressure function must be single-valued, we must take $c_2 = 0$.

Case (2): $\{c_1 = 0\}$.

Using $c_1 = 0$, we get $N_0 = N_1 = N_2 = N_3 = N_4 = N_5 = N_6 = 0$ and equation (6.156) reduces to a polynomial of degree six in $\gamma(\psi)$. Requiring this polynomial to hold true for all values of $\gamma(\psi)$, we must have

$$M_0 = M_1 = M_2 = M_3 = M_4 = M_5 = M_6 = 0$$

From $M_6 = 0$, we have

$$b_2 \left[-\frac{3}{2}b_1^2 + 6b_1b_3 - 3b_2^2 - \frac{\rho}{2\mu}b_1 \right] = 0 \tag{6.163}$$

Two cases arise: (a) $b_2 = 0$ and (b) $b_2 \neq 0$.

Subcase (a): $\{b_2 = 0\}$. Using $b_2 = 0$ in $M_1 = 0$, we get

$$b_3 \left[-12b_3^2 - 3b_1^2 + 12b_1b_3 + \frac{\rho}{\mu}(b_1 - 2b_3) \right] \tag{6.164}$$

Equation (6.164) implies that either $b_3 = 0$ or $b_3 \neq 0$.

Subsubcase (i): $\{b_3 = 0\}$. Employing $b_2 = b_3 = 0$ in $M_5 = 0$, we have

$$b_1 = -\frac{\rho}{3\mu} \tag{6.165}$$

Using (6.165) in (6.154) and integrating the resulting equation, we obtain

$$\gamma(\psi) = -\frac{2}{b_1\psi + b_4} \quad (6.166)$$

where b_4 is an arbitrary constant of integration and b_1 is given by equation (6.165).

Thus, $\frac{y}{x} = \text{constant}$ are possible streamlines for infinitely conducting MHD variably-inclined flow and the solutions are given by

$$\begin{aligned} u &= -\frac{6\mu x}{\rho y^2}, & v &= -\frac{6\mu}{\rho y} \\ \cot \theta &= -\frac{x}{y} + \frac{3\mu c_2}{\rho x y} \\ H &= \frac{\rho C y^2}{6\mu \sqrt{x^2 + y^2}} \csc \theta \\ \omega &= -\frac{12\mu x}{\rho y^3}, & \Omega &= -\frac{\rho C}{6\mu} \\ p &= \frac{3\mu^2}{\rho} \left[\frac{2x^2}{y^2} + \frac{6x^2}{y^4} \right] - \frac{\mu^* \rho^2 C^2}{72\mu^2} y^2 - \frac{\mu^* \rho C^2 c_2}{6\mu} \tan^{-1} \left(\frac{x}{y} \right) + p_0 \end{aligned} \quad (6.167)$$

where p_0 is an arbitrary constant. Since the pressure function must be single-valued, we must take $c_2 = 0$.

Subsubcase (ii): $\{b_3 \neq 0\}$. Multiplying $M_3 = 0$ by two and adding the result in equation (6.164), we get

$$b_1 = 2b_3 \quad (6.168)$$

Substituting $b_2 = 0$ and $b_1 = 2b_3$ in equation (6.154) and integrating the resulting equation, we find that

$$\gamma(\psi) = \tan(b_3\psi + b_5) \quad (6.169)$$

where b_5 is an arbitrary constant. In this case, the flow is irrotational with exact

solutions given by

$$\begin{aligned}
u &= \frac{x}{b_3(x^2 + y^2)}, & v &= \frac{y}{b_3(x^2 + y^2)} \\
\cot \theta &= \frac{c_2}{b_3(x^2 + y^2)} \\
H &= b_3 C \sqrt{x^2 + y^2} \csc \theta \\
\omega &= 0, & \Omega &= 2b_3 C \\
p &= -b_3 \mu^* C^2 x \sqrt{x^2 + y^2} + 2b_3 c_2 C^2 \left[\tan^{-1} \left(\frac{y}{x} \right) - b_3 \right] + p_0
\end{aligned} \tag{6.170}$$

where p_0 is an arbitrary constant of integration. The requirement that the pressure must be single-valued implies that $c_2 = 0$.

Subcase (b): $\{b_2 \neq 0\}$. Equation (6.163) and $M_0 = 0$ give

$$\begin{aligned}
b_1^2 &= 4b_1 b_3 - 2b_2^2 - \frac{\rho}{3\mu} b_1 \\
b_2^2 &= b_1 b_3 - 2b_3^2 - \frac{\rho}{3\mu} b_3
\end{aligned} \tag{6.171a,b}$$

Adding $M_0 = 0$ and $M_4 = 0$, we get

$$3b_1^2 - 12b_3^2 + \frac{\rho}{\mu} (b_1 - 2b_3) = 0 \tag{6.172}$$

Employing (6.171) in (6.172), we have

$$b_3 = 0$$

Using $b_3 = 0$, equation (6.171b) implies that $b_2 = 0$ which contradicts our assumption that $b_2 \neq 0$.

Theorem 6.7. *Streamline pattern $\frac{y}{x} = \text{constant}$ of steady plane motion is permissible for infinitely conducting MHD variably-inclined flow with solutions of rotational flow given by either equations (6.162) or (6.167) and irrotational flow given by (6.170).*

CHAPTER 7

FINITELY CONDUCTING TRANSVERSE-ALIGNED MHD CONFLUENT FLOWS

7.1 INTRODUCTION.

Confluent flows of an electrically conducting fluid of finite electrical conductivity in the presence of a transverse-aligned magnetic field are considered. Magnetohydrodynamic (MHD) plane flow is said to be *transverse-aligned* if the plane projection of the spatial magnetic field on the plane of flow is everywhere parallel to the planar velocity field. Grad [1960], Imai [1960, 1962] Hasimoto [1959] and Chandna and Chew [1980] studied transverse-aligned flows to investigate some of their basic characteristics.

Following the designation by Martin [1971], we shall call a steady plane flow of a viscous incompressible fluid *confluent* if any two of the following curves coincide in the physical plane:

- a) curves of constant pressure p or isobars,
- b) streamlines,
- c) curves of constant speed q or isovels,
- d) curves of constant energy h or isoenergetic lines,

e) curves of constant direction θ , where θ is the direction of flow in the physical plane,

f) curves of constant vorticity ω or isocurls.

Steady plane flows that are not confluent flows are called *fluent*.

Govindaraju [1972, 1981] studied six of these fifteen confluent flows. Furthermore, one of those confluent flows, called the Prandtl-Meyer flow, was investigated by Martin [1950] in one of his earlier works.

In the case of MHD flows, the curves of constant magnetic intensity magnitude H , the curves of constant current density Ω and the curves of constant magnetic energy become the seventh, eighth and ninth members of the families of curves above giving us a total of thirty-six confluent flows.

Chandna and Labropulu [1990], Labropulu and Chandna [1989, 1992] have already extended Martin's [1971] and Govindaraju's [1972, 1981] works and they have studied confluent flows for MHD aligned, MHD transverse-aligned and Cosserat fluid flow equations.

In this chapter, we study the following confluent flows:

(1) i) Isobars and isovels coincide with streamlines in the physical plane,
ii) Streamlines and isovels coincide with isobars and isocurls, respectively, in the physical plane,

(2) Isocurls coincide with streamlines in the physical plane.

7.2 FLGW EQUATIONS.

The steady plane flow of a viscous incompressible and electrically conducting fluid of finite electrical conductivity, in the presence of a magnetic field, is governed by equations (2.10) to (2.18).

Considering the flow to be transverse-aligned, we take

$$H_1 = \beta u, \quad H_2 = \beta v, \quad H_3 = H_3 \quad (7.1)$$

where $\beta(x, y)$ is some scalar function.

Introducing

$$h = \frac{1}{2}\rho q^2 + p + \frac{1}{2}\mu^* H^2 \quad (7.2)$$

where $q = \sqrt{u^2 + v^2}$ is the speed and $H = \sqrt{H_1^2 + H_2^2 + H_3^2}$ is the magnetic intensity magnitude and using (7.1), equations (2.10) to (2.18) can be replaced by:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (\text{continuity}) \quad (7.3)$$

$$\frac{\partial h}{\partial x} + \mu \frac{\partial \omega}{\partial y} - \rho v \omega + \mu^* \beta v \Omega = \frac{1}{2} \mu^* \frac{\partial(\beta^2 q^2)}{\partial x} \quad (x - \text{momentum}) \quad (7.4)$$

$$\frac{\partial h}{\partial y} - \mu \frac{\partial \omega}{\partial x} + \rho u \omega - \mu^* \beta u \Omega = \frac{1}{2} \mu^* \frac{\partial(\beta^2 q^2)}{\partial y} \quad (y - \text{momentum}) \quad (7.5)$$

$$u \frac{\partial H_3}{\partial x} + v \frac{\partial H_3}{\partial y} = 0 \quad (z - \text{momentum}) \quad (7.6)$$

$$\Omega = -\mu^* \sigma C = C_0 \quad (\text{say}) \quad (\text{diffusion}) \quad (7.7)$$

$$\frac{\partial^2 H_3}{\partial x^2} + \frac{\partial^2 H_3}{\partial y^2} = 0 \quad (\text{diffusion}) \quad (7.8)$$

$$u \frac{\partial \beta}{\partial x} + v \frac{\partial \beta}{\partial y} = 0 \quad (\text{solenoidal}) \quad (7.9)$$

$$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (\text{vorticity}) \quad (7.10)$$

$$\Omega = \beta \omega + v \frac{\partial \beta}{\partial x} - u \frac{\partial \beta}{\partial y} \quad (\text{current density}) \quad (7.11)$$

This is a system of nine equations in seven unknowns u , v , h , H_3 , ω , Ω , and β as functions of x , y . If we take $H_3 = 0$ in the above system of equations, we obtain the governing equations of aligned MHD flow.

7.3 ALTERNATE FORMS OF EQUATIONS OF MOTION.

Equation of continuity (7.3) implies the existence of a streamfunction $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x} = -v, \quad \frac{\partial \psi}{\partial y} = u \quad (7.12)$$

Having recorded the results from differential geometry in section 2.2, we follow and employ Martin's [1971] pioneering work and transform equations (7.3) to (7.11) into new forms with new variables ϕ, ψ .

7.3.1 First Form.

Linear Momentum Equations: We employ (7.12) in the linear momentum equations (7.4) to (7.6) and we make use of (2.25) and (7.7) to obtain

$$\begin{aligned} \frac{\partial h}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial y}{\partial \phi} + \mu \left(-\frac{\partial \omega}{\partial \phi} \frac{\partial x}{\partial \psi} + \frac{\partial \omega}{\partial \psi} \frac{\partial x}{\partial \phi} \right) - (\rho\omega - \mu^* \beta C_0) \frac{\partial y}{\partial \phi} \\ = \frac{1}{2} \left[\frac{\partial (\beta^2 q^2)}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial (\beta^2 q^2)}{\partial \psi} \frac{\partial y}{\partial \phi} \right] \end{aligned} \quad (7.13)$$

$$\begin{aligned} \frac{\partial h}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial h}{\partial \psi} \frac{\partial x}{\partial \phi} + \mu \left(\frac{\partial \omega}{\partial \phi} \frac{\partial y}{\partial \psi} - \frac{\partial \omega}{\partial \psi} \frac{\partial y}{\partial \phi} \right) - (\rho\omega - \mu^* \beta C_0) \frac{\partial x}{\partial \phi} \\ = \frac{1}{2} \left[\frac{\partial (\beta^2 q^2)}{\partial \phi} \frac{\partial x}{\partial \psi} - \frac{\partial (\beta^2 q^2)}{\partial \psi} \frac{\partial x}{\partial \phi} \right] \end{aligned} \quad (7.14)$$

$$\frac{\partial H_3}{\partial \phi} = 0, \quad \text{or} \quad H_3 = H_3(\psi) \quad (7.15)$$

Multiplying (7.13) by $\frac{\partial y}{\partial \psi}$, (7.14) by $\frac{\partial x}{\partial \psi}$ and adding, we get

$$G \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] - F \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] = -\mu J \frac{\partial \omega}{\partial \psi}$$

Likewise, multiplying (7.13) by $\frac{\partial y}{\partial \phi}$, (7.14) by $\frac{\partial x}{\partial \phi}$ and adding, we have

$$-F \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] + E \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] = \mu J \frac{\partial \omega}{\partial \phi}$$

Diffusion Equation (7.8): Transforming equation (7.8) to (ϕ, ψ) -net, we obtain

$$\frac{\partial}{\partial \phi} \left(\frac{G}{J} \frac{\partial H_3}{\partial \phi} - \frac{F}{J} \frac{\partial H_3}{\partial \psi} \right) + \frac{\partial}{\partial \psi} \left(-\frac{F}{J} \frac{\partial H_3}{\partial \phi} + \frac{E}{J} \frac{\partial H_3}{\partial \psi} \right) = 0 \quad (7.16)$$

Substitution of (7.15) in (7.16) yields

$$\frac{\partial}{\partial \phi} \left(\frac{F}{J} \frac{dH_3}{d\psi} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{dH_3}{d\psi} \right) = 0$$

Solenoidal Equation: Using (7.12) in (7.9) and transforming to (ϕ, ψ) -net, we get

$$\frac{\partial \psi}{\partial y} \left(\frac{\partial \beta}{\partial \phi} \frac{\partial \phi}{\partial x} + \frac{\partial \beta}{\partial \psi} \frac{\partial \psi}{\partial x} \right) - \frac{\partial \psi}{\partial x} \left(\frac{\partial \beta}{\partial \phi} \frac{\partial \phi}{\partial y} + \frac{\partial \beta}{\partial \psi} \frac{\partial \psi}{\partial y} \right) = 0$$

which gives

$$\frac{\partial \beta}{\partial \phi} = 0 \quad \text{or} \quad \beta = \beta(\psi) \quad (7.17)$$

Current Density Equation: Employing (7.12) in (7.11), transforming to (ϕ, ψ) -net and using (2.25) and (2.24), we have

$$C_0 = \beta \omega + \frac{F}{J^2} \frac{\partial \beta}{\partial \phi} - \frac{E}{J^2} \frac{\partial \beta}{\partial \psi} \quad (7.18)$$

Substitution of (7.17) in (7.18) yields

$$C_0 = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi}$$

Equations of Continuity and Vorticity: Martin [1971] obtained the necessary and sufficient condition for the flow of a fluid along the coordinate lines $\psi = \text{constant}$ of a curvilinear system (2.22), with ds^2 given by (2.23), to satisfy the principle of conservation of mass to be

$$Wq = \sqrt{E}, \quad u + iv = \frac{\sqrt{E}}{W} e^{i\alpha}$$

He has also proven that

$$\omega = \frac{1}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right]$$

In this work, we consider that the fluid flows along a streamline towards higher parameter values of ϕ so that $J = W > 0$.

Summing up, we have the following theorem:

Theorem 7.1. *If the streamlines $\psi(x, y) = \text{constant}$ and an arbitrary family of curves $\phi(x, y) = \text{constant}$ generate a curvilinear net in the physical plane of a viscous incompressible and electrically conducting fluid, then transverse-aligned flow in independent variables ϕ, ψ is governed by the system:*

$$q = \frac{\sqrt{E}}{J} \quad (7.19)$$

$$G \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] - F \left[\frac{\partial h}{\partial \psi} + \rho \omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] = -\mu J \frac{\partial \omega}{\partial \psi} \quad (7.20)$$

$$-F \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] + E \left[\frac{\partial h}{\partial \psi} + \rho \omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] = \mu J \frac{\partial \omega}{\partial \phi} \quad (7.21)$$

$$\omega = \frac{1}{J} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{J} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \right) \right] \quad (7.22)$$

$$C_0 = \beta \omega - \frac{E}{J^2} \frac{d\beta}{d\psi} \quad (7.23)$$

$$\frac{\partial}{\partial \psi} \left(\frac{J}{E} \Gamma_{11}^2 \right) - \frac{\partial}{\partial \phi} \left(\frac{J}{E} \Gamma_{12}^2 \right) = 0 \quad (7.24)$$

$$\frac{\partial}{\partial \phi} \left(\frac{F}{J} \frac{dH_3}{d\psi} \right) - \frac{\partial}{\partial \psi} \left(\frac{E}{J} \frac{dH_3}{d\psi} \right) = 0 \quad (7.25)$$

of seven equations in eight unknowns $E, F, G, \omega, q, h, \beta$ and H_3 as functions of ϕ, ψ .

Furthermore, having determined a solution of (7.19) to (7.25), we find x, y as functions of ϕ, ψ from

$$z = x + iy = \int \frac{1}{\sqrt{E}} e^{i\alpha} [E d\phi + (F + iJ) d\psi] \quad (7.26)$$

where

$$\alpha = \int \frac{J}{E} [\Gamma_{11}^2 d\phi + \Gamma_{12}^2 d\psi] \quad (7.27)$$

and u, v from

$$u + iv = \frac{\sqrt{E}}{J} e^{i\alpha} \quad (7.28)$$

to complete the solution.

7.3.2 Second Form (Bar-System).

The development of the bar-system for our fluid flow in this subsection follows Martin's derivation of the bar-system for the Navier-Stokes equations.

Linear Momentum Equations: We consider one more equivalent form of the momentum equations (7.20) and (7.21). We multiply (7.20) by E , (7.21) by F and add to obtain an equation which along with (7.21) forms the set

$$F\mu \frac{\partial \omega}{\partial \phi} - E\mu \frac{\partial \omega}{\partial \psi} = J \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] \quad (7.29)$$

$$-F \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] + E \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] = \mu J \frac{\partial \omega}{\partial \phi} \quad (7.30)$$

Solving these two equations for E and F , we find that

$$\begin{aligned} E \left\{ \mu \frac{\partial \omega}{\partial \phi} \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] - \mu \frac{\partial \omega}{\partial \psi} \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] \right\} \\ = J \left\{ \mu^2 \left(\frac{\partial \omega}{\partial \phi} \right)^2 + \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right]^2 \right\} \end{aligned} \quad (7.31)$$

$$\begin{aligned} F \left\{ \mu \frac{\partial \omega}{\partial \phi} \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] - \mu \frac{\partial \omega}{\partial \psi} \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] \right\} \\ = J \left\{ \mu^2 \frac{\partial \omega}{\partial \phi} \frac{\partial \omega}{\partial \psi} + \left[\frac{\partial h}{\partial \phi} - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] \left[\frac{\partial h}{\partial \psi} + \rho\omega - \mu^* \beta C_0 - \frac{1}{2}\mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] \right\} \end{aligned} \quad (7.32)$$

We define

$$\begin{aligned}
\tilde{E} &= \mu^2 \left(\frac{\partial \omega}{\partial \phi} \right)^2 + \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right]^2, \\
\tilde{F} &= \mu^2 \frac{\partial \omega}{\partial \phi} \frac{\partial \omega}{\partial \psi} + \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right] \left[\frac{\partial h}{\partial \psi} + \rho \omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right], \\
\tilde{G} &= \mu^2 \left(\frac{\partial \omega}{\partial \psi} \right)^2 + \left[\frac{\partial h}{\partial \psi} + \rho \omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right]^2, \\
\tilde{J} &= \mu \frac{\partial \omega}{\partial \phi} \left[\frac{\partial h}{\partial \psi} + \rho \omega - \mu^* \beta C_0 - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \psi} \right] - \mu \frac{\partial \omega}{\partial \psi} \left[\frac{\partial h}{\partial \phi} - \frac{1}{2} \mu^* \frac{\partial (\beta^2 q^2)}{\partial \phi} \right],
\end{aligned} \tag{7.33}$$

and

$$\tilde{W} = \sqrt{\tilde{E}\tilde{G} - \tilde{F}^2}$$

Using (7.33), equations (7.31) and (7.32) can be written as

$$\frac{E}{J} = \frac{\tilde{E}}{\tilde{J}}, \quad \frac{F}{J} = \frac{\tilde{F}}{\tilde{J}} \tag{7.34}$$

and

$$\tilde{J} = \pm \tilde{W} \tag{7.35}$$

Since $E > 0$ and $\tilde{E} > 0$, the first equation in (7.34) shows that J and \tilde{J} have the same sign, so that $\tilde{J} = \pm \tilde{W}$ accordingly as $J = \pm W$. Consequently, (7.34) can be written as

$$\frac{E}{W} = \frac{\tilde{E}}{\tilde{W}}, \quad \frac{F}{W} = \frac{\tilde{F}}{\tilde{W}} \tag{7.36}$$

Conversely, these equations imply the linear momentum equations (7.31), (7.30) and are, therefore, equivalent to them. Therefore, the linear momentum equations of an incompressible and electrically conducting transverse-aligned MHD fluid flow in the form (7.29), (7.30) are equivalent to (7.36) where \tilde{E} , \tilde{F} , \tilde{G} , \tilde{J} and \tilde{W} are defined in (7.33).

Letting k be the kinetic energy, we have

$$k = \frac{1}{2} \rho q^2 = h - p - \frac{1}{2} \mu^* H^2 = \frac{1}{2} \rho \frac{E}{W^2}$$

From this, we obtain

$$W = \frac{\rho E}{2k\bar{W}} \quad (7.37)$$

Using (7.36) in (7.37), we get

$$W = \frac{\rho}{2k} \frac{\tilde{E}}{\tilde{W}} \quad (7.38)$$

so that the common ratio in (7.36) can be evaluated as

$$\frac{E}{\tilde{E}} = \frac{F}{\tilde{F}} = \frac{G}{\tilde{G}} = \frac{W}{\tilde{W}} = \frac{\rho \tilde{E}}{2k\tilde{W}^2} \quad (7.39)$$

or in the form

$$E = \frac{\rho \tilde{E}^2}{2k\tilde{W}^2}, \quad F = \frac{\rho \tilde{E}\tilde{F}}{2k\tilde{W}^2}, \quad G = \frac{\rho \tilde{E}\tilde{G}}{2k\tilde{W}^2} \quad (7.40)$$

In view of the definitions of \tilde{E} , \tilde{F} , \tilde{G} and \tilde{W} in (7.33), we note that E , F and G become known functions of ϕ , ψ as long as ω , k and p are established as functions of ϕ , ψ .

We consider that $\tilde{J} = \tilde{W}$ and $J = W$ as we have assumed that our fluid flows towards higher values of ϕ along a streamline.

Vorticity Equation: Using (7.39), equation (7.22) becomes

$$\frac{\rho \tilde{E} \omega}{2k\tilde{J}} = \frac{\partial}{\partial \phi} \left(\frac{\tilde{F}}{\tilde{J}} \right) - \frac{\partial}{\partial \psi} \left(\frac{\tilde{E}}{\tilde{J}} \right)$$

Gauss Equation: Using $h = \frac{1}{2}\rho q^2 + p + \frac{1}{2}\mu^* H^2 = \frac{\rho E}{2W^2} + p + \frac{1}{2}\mu^* H^2$ in equations (7.20) and (7.21), we have

$$\begin{aligned} G \frac{\partial}{\partial \phi} \left(\frac{\rho E}{2W^2} + p + \frac{1}{2}\mu^* H_3^2 \right) - F \left\{ \frac{\partial}{\partial \psi} \left(\frac{\rho E}{2W^2} + p + \frac{1}{2}\mu^* H_3^2 \right) + \frac{\rho}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \right. \right. \\ \left. \left. - \frac{\partial}{\partial \psi} \left(\frac{E}{W} \right) \right] - \mu^* \beta C_0 \right\} = -\mu J \frac{\partial \omega}{\partial \psi} \end{aligned} \quad (7.41)$$

$$-F \frac{\partial}{\partial \phi} \left(\frac{\rho E}{2W^2} + p + \frac{1}{2}\mu^* H_3^2 \right) + E \left\{ \frac{\partial}{\partial \psi} \left(\frac{\rho E}{2W^2} + p + \frac{1}{2}\mu^* H_3^2 \right) + \frac{\rho}{W} \left[\frac{\partial}{\partial \phi} \left(\frac{F}{W} \right) - \right.$$

$$\left. -\frac{\partial}{\partial\psi} \left(\frac{E}{W} \right) \right] - \mu^* \beta C_0 \Big\} = \mu J \frac{\partial\omega}{\partial\phi} \quad (7.42)$$

Employing (2.39) to (2.41), equations (7.41) and (7.42) become

$$\begin{aligned} G \frac{\partial p}{\partial\phi} - \rho \Gamma_{12}^2 - F \frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \mu^* \beta C_0 &= -\mu J \frac{\partial\omega}{\partial\psi} \\ -F \frac{\partial p}{\partial\phi} + \rho \Gamma_{11}^2 + E \frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 &= \mu J \frac{\partial\omega}{\partial\phi} \end{aligned}$$

Solving for Γ_{11}^2 and Γ_{12}^2 , we get

$$\begin{aligned} \Gamma_{11}^2 &= \frac{J}{\rho} \left\{ \mu \frac{\partial\omega}{\partial\phi} + \frac{F}{J} \frac{\partial p}{\partial\phi} - \frac{E}{J} \frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \frac{E}{J} \mu^* \beta C_0 \right\} \\ \Gamma_{12}^2 &= \frac{J}{\rho} \left\{ \mu \frac{\partial\omega}{\partial\psi} + \frac{G}{J} \frac{\partial p}{\partial\phi} - \frac{F}{J} \frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \frac{F}{J} \mu^* \beta C_0 \right\} \end{aligned} \quad (7.43)$$

Using these expressions for Γ_{11}^2 and Γ_{12}^2 and (7.36) in Gauss equation (7.24), we obtain after some simplification

$$\begin{aligned} \frac{\partial}{\partial\psi} \left\{ \frac{\tilde{F}}{\tilde{J}} \frac{1}{k} \frac{\partial p}{\partial\phi} - \frac{\tilde{E}}{\tilde{J}} \frac{1}{k} \left[\frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} - \frac{\partial}{\partial\phi} \left\{ \frac{\tilde{G}}{\tilde{J}} \frac{1}{k} \frac{\partial p}{\partial\phi} - \right. \\ \left. - \frac{\tilde{F}}{\tilde{J}} \frac{1}{k} \left[\frac{\partial}{\partial\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} = \mu \frac{\partial \left(\frac{1}{k}, \omega \right)}{\partial (\phi, \psi)} \end{aligned}$$

Equations (7.23) and (7.25): Substituting (7.36) in (7.23) and (7.25), we get

$$\tilde{\omega}_J = \beta\omega - \frac{2}{\rho} k \frac{d\beta}{d\psi}$$

$$\frac{\partial}{\partial\phi} \left(\frac{\tilde{F}}{\tilde{J}} \frac{dH_3}{d\psi} \right) - \frac{\partial}{\partial\psi} \left(\frac{\tilde{E}}{\tilde{J}} \frac{dH_3}{d\psi} \right) = 0$$

Summing up the above results, we have the following theorem:

Theorem 7.2. *When the streamlines, $\psi(x, y) = \text{constant}$, in the presence of a transverse-aligned magnetic field in a flow of an incompressible and electrically conducting fluid are taken as one set of coordinate lines in a curvilinear coordinate*

system ϕ, ψ in the physical plane, then the system (7.19) to (7.25) may be replaced by the following system:

$$\frac{\partial}{\partial \phi} \left(\frac{\tilde{F}}{\tilde{J}} \right) - \frac{\partial}{\partial \psi} \left(\frac{\tilde{E}}{\tilde{J}} \right) = \frac{\rho \omega}{2k} \frac{\tilde{E}}{\tilde{J}} \quad (7.44)$$

$$\begin{aligned} \frac{\partial}{\partial \psi} \left\{ \frac{\tilde{F}}{\tilde{J}} \frac{1}{k} \frac{\partial p}{\partial \phi} - \frac{\tilde{E}}{\tilde{J}} \frac{1}{k} \left[\frac{\partial}{\partial \psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} - \frac{\partial}{\partial \phi} \left\{ \frac{\tilde{G}}{\tilde{J}} \frac{1}{k} \frac{\partial p}{\partial \phi} - \right. \\ \left. - \frac{\tilde{F}}{\tilde{J}} \frac{1}{k} \left[\frac{\partial}{\partial \psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} = \mu \frac{\partial \left(\frac{1}{k}, \omega \right)}{\partial (\phi, \psi)} \end{aligned} \quad (7.45)$$

$$C_0 = \beta \omega - \frac{2}{\rho} k \frac{d\beta}{d\psi} \quad (7.46)$$

$$\frac{\partial}{\partial \phi} \left(\frac{\tilde{F}}{\tilde{J}} \frac{dH_3}{d\psi} \right) - \frac{\partial}{\partial \psi} \left(\frac{\tilde{E}}{\tilde{J}} \frac{dH_3}{d\psi} \right) = 0 \quad (7.47)$$

of four equations for five unknowns ω, k, p, β and H_3 as functions of ϕ, ψ . Here $\tilde{E}, \tilde{F}, \tilde{G}$ and \tilde{J} are given by (7.33) and k is the kinetic energy.

Given a solution

$$\omega = \omega(\phi, \psi), \quad k = k(\phi, \psi), \quad p = p(\phi, \psi), \quad \beta = \beta(\psi), \quad H_3 = H_3(\psi)$$

for these equations, the flow in the physical plane is obtained from

$$\begin{aligned} z = \pm \int e^{i\alpha} \sqrt{\frac{\rho}{2k}} \left[\frac{\tilde{E}}{\tilde{J}} d\phi + \left(\frac{\tilde{F}}{\tilde{J}} + i \right) d\psi \right] \\ \alpha = \int \left(\frac{\partial \alpha}{\partial \phi} d\phi + \frac{\partial \alpha}{\partial \psi} d\psi \right) \end{aligned} \quad (7.48)$$

with $\frac{\partial \alpha}{\partial \phi}$ and $\frac{\partial \alpha}{\partial \psi}$ given by

$$\begin{aligned} \frac{\partial \alpha}{\partial \phi} = \frac{1}{2k} \left\{ \mu \frac{\partial \omega}{\partial \phi} + \frac{\tilde{F}}{\tilde{J}} \frac{\partial p}{\partial \phi} - \frac{\tilde{E}}{\tilde{J}} \left[\frac{\partial}{\partial \psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} \\ \frac{\partial \alpha}{\partial \psi} = \frac{1}{2k} \left\{ \mu \frac{\partial \omega}{\partial \psi} + \frac{\tilde{G}}{\tilde{J}} \frac{\partial p}{\partial \phi} - \frac{\tilde{F}}{\tilde{J}} \left[\frac{\partial}{\partial \psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} \end{aligned} \quad (7.49)$$

and the flow in the hodograph plane is given by

$$u + iv = \sqrt{\frac{2k}{\rho}} e^{i\alpha} \quad (7.50)$$

to complete the solution.

The underdetermined system (7.44) to (7.47) can be made determinate by specifying the choice of the ϕ -curves.

In the subsequent sections, we study the applications of the above results by considering flows in which certain conditions are placed *a priori*.

7.4 STREAMLINES AND ISOBARS COINCIDE.

Govindaraju [1981] investigated non-MHD viscous fluid flows when streamlines and isobars coincide under the additional assumption that speed q is constant on streamlines.

In this section, we shall consider two cases. In the first case, along with our assumption $p = p(\psi)$, we shall make an additional assumption that the speed is constant on streamlines, i.e. $q = q(\psi)$. In the second case, we shall assume that the speed remains constant on the isocurls, i.e. $q = q(\omega)$.

7.4.1 Speed Constant on Streamlines.

We assume that $q = q(\psi)$, $q'(\psi) \neq 0$ and $p = p(\psi)$. Since $k = \frac{1}{2}\rho q^2$, then $k = k(\psi)$ also. Using these two assumptions in (7.33), we get

$$\begin{aligned}\tilde{E} &= \mu^2 \left(\frac{\partial \omega}{\partial \phi} \right)^2 \\ \tilde{F} &= \mu^2 \frac{\partial \omega}{\partial \phi} \frac{\partial \omega}{\partial \psi} \\ \tilde{G} &= \mu^2 \left(\frac{\partial \omega}{\partial \psi} \right)^2 + \left[k' + p' + \rho \omega - \mu^* \beta C_0 + \frac{1}{2} (H_3^2)' \right]^2 \\ \tilde{J} &= \mu \frac{\partial \omega}{\partial \phi} \left[k' + p' + \rho \omega - \mu^* \beta C_0 + \frac{1}{2} (H_3^2)' \right]\end{aligned}\tag{7.51}$$

where prime denotes differentiation with respect to ψ .

Using equations (7.51) in (7.44), we obtain

$$\begin{aligned}\frac{\partial}{\partial \phi} \left[\frac{\mu \frac{\partial \omega}{\partial \psi}}{k' + p' + \rho \omega - \mu^* \beta C_0 + \frac{1}{2} (H_3^2)'} \right] - \frac{\partial}{\partial \psi} \left[\frac{\mu \frac{\partial \omega}{\partial \phi}}{k' + p' + \rho \omega - \mu^* \beta C_0 + \frac{1}{2} (H_3^2)'} \right] \\ = \frac{\rho \omega}{2k} \frac{\mu \frac{\partial \omega}{\partial \phi}}{k' + p' + \rho \omega - \mu^* \beta C_0 + \frac{1}{2} (H_3^2)'}\end{aligned}$$

which after simplification gives

$$\frac{(k + p + \frac{1}{2} \mu^* H_3^2)'' - \mu^* \beta' C_0}{(k + p + \frac{1}{2} \mu^* H_3^2)' + \rho \omega - \mu^* \beta C_0} = \frac{\rho \omega}{2k}\tag{7.52}$$

Since $k = k(\psi)$, $p = p(\psi)$, $H_3 = H_3(\psi)$ and $\beta = \beta(\psi)$, then equation (7.52) implies that $\omega = \omega(\psi)$ and therefore (7.51) yields $\tilde{E} = \tilde{J} = 0$ so that the bar-system fails.

We shall now use equations (7.19) to (7.25) and take the curvilinear coordinate net (ϕ, ψ) to be an orthogonal net, so that $F = 0$.

Using $k = k(\psi)$ and $p = p(\psi)$, equation (7.20) implies that $\omega = \omega(\phi)$ and (7.21) gives

$$\frac{E}{J} = \frac{\mu \frac{d\omega}{d\phi}}{(k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0} \quad (7.53)$$

Using (7.53) in (7.22), we have

$$\omega = \frac{\mu \frac{d\omega}{d\phi}}{J} \frac{(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0}{\left\{ (k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0 \right\}^2} \quad (7.54)$$

Employing (7.53) and $J^2 = EG$ in (7.54), we get

$$\omega = \frac{(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0}{(k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0} \frac{1}{G} \quad (7.55)$$

If $(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0 = 0$, then $\omega = 0$. Thus, we assume that $(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0 \neq 0$ and use (7.53) and (7.55) to obtain

$$\begin{aligned} E &= \frac{\mu^2 \left(\frac{d\omega}{d\phi} \right)^2 \left[(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0 \right]}{\omega \left[(k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0 \right]} \\ G &= \frac{\mu \frac{d\omega}{d\phi} \left[(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0 \right]}{\omega \left[(k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0 \right]} \end{aligned} \quad (7.56)$$

and

$$J = \sqrt{EG} = \frac{\mu \frac{d\omega}{d\phi} \left[(k + p + \frac{1}{2}\mu^* H_3^2)'' - \mu^* \beta' C_0 \right]}{\omega \left[(k + p + \frac{1}{2}\mu^* H_3^2)' + \rho\omega - \mu^* \beta C_0 \right]}$$

Substitution of (7.56) in (7.24) yields after some calculations

$$\frac{d\omega}{d\phi} [b_0 \omega^4 + b_1 \omega^3 + b_2 \omega^2 + b_3 \omega + b_4] = 0 \quad (7.57)$$

where b_i , $i = 0, 1, 2, 3$ are functions of ψ and are given by

$$b_0 = \mu \rho^2 \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)^{(iv)} - \mu^* \beta''' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right] \\ - \mu \rho^2 \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)''' - \mu^* \beta'' C_0 \right]$$

$$b_1 = 2\mu \rho \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)^{(iv)} - \mu^* \beta''' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right] \\ * \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right] \\ - 2\mu \rho \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right] \\ - 4\mu \rho \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)''' - \mu^* \beta'' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^2 \\ - \frac{\rho^3}{\mu} \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^2 \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right]$$

$$b_2 = \mu \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)^{(iv)} - \mu^* \beta''' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right] \\ * \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right] \\ - \mu \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)''' - \mu^* \beta'' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right]^2 \\ - 4\mu \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)''' - \mu^* \beta'' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^2 \\ * \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right] \\ - \frac{3\rho^2}{\mu} \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right]^2 \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^2 \\ + 6\mu \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^4$$

$$b_3 = -\frac{3\rho}{\mu} \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right]^3 \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right]^2$$

$$b_4 = -\frac{1}{\mu} \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 \right] \left[\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)' - \mu^* \beta C_0 \right]$$

From equation (7.57), we note that either

$$\frac{d\omega}{d\phi} = 0$$

or

$$b_0 \omega^4 + b_1 \omega^3 + b_2 \omega^2 + b_3 \omega + b_4 = 0$$

The second equation gives $\omega = \omega(\psi)$ unless the coefficients b_i , $i = 0, 1, 2, 3, 4$ vanish simultaneously. In particular, $b_4 = 0$ implies that

$$\left(k + p + \frac{1}{2} \mu^* H_3^2 \right)'' - \mu^* \beta' C_0 = 0$$

contrary to our hypothesis. Hence, ω is a constant everywhere.

Employing (7.10) and $\omega = \omega_0$ in (7.58), we get

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega_0 \quad (7.58)$$

where ω_0 is an arbitrary constant.

We introduce the complex variables

$$z = x + iy, \quad \bar{z} = x - iy$$

where $i = \sqrt{-1}$ and we find that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad \frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right) \quad (7.59)$$

Using (7.59) in (7.58), we obtain

$$4 \frac{\partial^2 \psi}{\partial z \partial \bar{z}} = -\omega_0$$

which gives

$$\psi(x, y) = -\frac{\omega_0}{4} (x^2 + y^2) + A(x, y) \quad (7.60)$$

where $A(x, y)$ is a harmonic function.

Summing up, we have:

Theorem 7.3. *If the streamlines and isobars coincide and the kinetic energy is constant on streamlines in a steady two-dimensional transverse-aligned flow of a viscous incompressible and electrically conducting fluid, the vorticity is constant and the streamfunction $\psi(x, y)$ is given by (7.60).*

7.4.2 Speed Constant on Isocurls.

In this case, along with our assumption that the streamlines and isobars coincide, we also assume that the speed is constant on isocurls, i.e. $q = q(\omega)$. Since $k = \frac{1}{2}\rho q^2$, then $k = k(\omega)$ also. We choose the (ϕ, ψ) coordinate net to be an orthogonal net so that (7.32) for the assumed flows gives

$$\bar{F} = \frac{\partial \omega}{\partial \phi} \left\{ (\mu^2 + k'^2) \frac{\partial \omega}{\partial \psi} + k' \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right] \right\} = 0 \quad (7.61)$$

where prime denotes differentiation with respect to ω .

Using (7.61) and our assumptions in equations (7.44) to (7.47), we find that the equations governing such flows are

$$-\frac{\partial}{\partial \psi} \left(\frac{\bar{E}}{\bar{J}} \right) = \frac{\rho \omega}{2k} \frac{\bar{E}}{\bar{J}} \quad (7.62)$$

$$\frac{\partial}{\partial \psi} \left\{ \frac{\bar{E}}{\bar{J}k} \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \right\} = 0 \quad (7.63)$$

$$\beta \omega - \frac{2}{\rho} k \frac{d\beta}{d\psi} = C_0 \quad (7.64)$$

$$\frac{\partial}{\partial \psi} \left(\frac{\bar{E}}{\bar{J}} \frac{dH_3}{d\psi} \right) = 0 \quad (7.65)$$

along with (7.61). Here

$$\bar{E} = (\mu^2 + k'^2) \left(\frac{\partial \omega}{\partial \phi} \right)^2, \quad \bar{J} = \mu \frac{\partial \omega}{\partial \phi} \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right] \quad (7.66)$$

When $\omega = \omega(\psi)$ and therefore $k = k(\psi)$, then the vorticity remains constant everywhere as we have already proven in section 7.4.1. Thus, we assume that $\omega \neq \omega(\psi)$, that is $\frac{\partial \omega}{\partial \phi} \neq 0$.

Solving equation (7.61) for $\frac{\partial \omega}{\partial \psi}$, we get

$$\frac{\partial \omega}{\partial \psi} = - \frac{k' \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right]}{\mu^2 + k'^2} \quad (7.67)$$

Integrating (7.63) with respect to ψ , we obtain

$$\frac{(\mu^2 + k'^2) \frac{\partial \omega}{\partial \phi} \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right]}{\mu k \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right]} = g(\phi) \quad (7.68)$$

where $g(\phi)$ is an arbitrary function of ϕ .

Assuming that $\left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \neq 0$, equation (7.68) gives

$$\frac{\partial \omega}{\partial \phi} = \frac{\mu k g(\phi) \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right]}{(\mu^2 + k'^2) \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right]} \quad (7.69)$$

Using (7.68) and (7.69) in the integrability condition $\frac{\partial^2 \omega}{\partial \phi \partial \psi} = \frac{\partial^2 \omega}{\partial \psi \partial \phi}$, we get after some tedious calculations

$$\begin{aligned} (kk'' - k'^2) \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right]^2 \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \\ - \rho \omega k (\mu^2 + k'^2) \left[\frac{d^2}{d\psi^2} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* C_0 \frac{d\beta}{d\psi} \right] = 0 \end{aligned} \quad (7.70)$$

On the other hand, employing (7.66), (7.67) and (7.69), equation (7.62) reduces to

$$\begin{aligned} 2k'^2 \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho \omega - \mu^* \beta C_0 \right] \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] \\ + 2k (\mu^2 + k'^2) \left[\frac{d^2}{d\psi^2} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* C_0 \frac{d\beta}{d\psi} \right] \\ - \rho \omega (\mu^2 + k'^2) \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] = 0 \end{aligned} \quad (7.71)$$

Eliminating $\left[\frac{d^2}{d\psi^2} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* C_0 \frac{d\beta}{d\psi} \right]$ from (7.70) and (7.71), we have

$$2(kk'' - k'^2) \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho\omega - \mu^* \beta C_0 \right]^2 + 2\rho\omega k'^2 \left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho\omega - \mu^* \beta C_0 \right] - \rho^2 \omega^2 (\mu^2 + k'^2) = 0 \quad (7.72)$$

Equation (7.72) is a quadratic in $\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho\omega - \mu^* \beta C_0$ whose coefficients are functions of ω and clearly is not an identity because $\omega \neq 0$, $k' \neq 0$ and $\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho\omega - \mu^* \beta C_0 \neq 0$. Hence, we can conclude that $\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) + \rho\omega - \mu^* \beta C_0$ must be a function of ω . Since $p = p(\psi)$, $\beta = \beta(\psi)$, $H_3 = H_3(\psi)$ and $\omega = \omega(\phi, \psi)$, then we must have

$$\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 = \text{constant} = a \quad (\text{say}) \quad (7.73)$$

and equation (7.70) gives

$$(kk'' - k'^2) [a + \rho\omega]^2 a = 0$$

If $a + \rho\omega = 0$, then $\frac{\partial\omega}{\partial\phi} = 0$ contrary to our assumption. Thus $a + \rho\omega \neq 0$ and the above equation yields

$$kk'' - k'^2 = 0$$

which upon integration gives

$$k = k_0 e^{b\omega}$$

where k_0 and b are arbitrary constants.

Using this value of k in (7.72), we obtain

$$2\rho k_0^2 b^2 (a + \rho\omega) e^{2b\omega} - \rho^2 \omega (\mu^2 + k_0^2 b^2 e^{2b\omega}) = 0$$

or

$$k_0^2 b^2 (2a + \rho\omega) e^{2b\omega} - \rho\mu^2 \omega = 0$$

which implies that ω is a constant. But this contradicts our assumption that $\frac{\partial \omega}{\partial \phi} \neq 0$. Hence, $\frac{d}{d\psi} (p + \frac{1}{2} \mu^* H_3^2) - \mu^* \beta C_0 = 0$.

Summing up, we have:

Theorem 7.4. *If the streamlines and isobars coincide and the speed remains constant along the isocurls in a steady two-dimensional transverse-aligned flow of an incompressible and electrically conducting viscous fluid, then either (i) the vorticity is constant everywhere, or (ii) $p = p(\psi)$ such that*

$$\frac{dp}{d\psi} = -\frac{1}{2} \mu^* \frac{dH_3^2}{d\psi} + \mu^* \beta C_0.$$

If $\frac{d}{d\psi} (p + \frac{1}{2} \mu^* H_3^2) - \mu^* \beta C_0 = 0$, then Gauss equation (7.63) is identically satisfied and equation (7.62) gives

$$(\mu^2 + k'^2) \frac{\partial^2 \omega}{\partial \phi \partial \psi} + 2k'k'' \frac{\partial \omega}{\partial \phi} \frac{\partial \omega}{\partial \psi} + \rho \left[k' + \frac{\omega}{2k} (\mu^2 + k'^2) \right] \frac{\partial \omega}{\partial \phi} = 0$$

Using (7.67) to eliminate $\frac{\partial^2 \omega}{\partial \phi \partial \psi}$ and $\frac{\partial \omega}{\partial \psi}$ from the above equation, we get

$$2kk'' = \mu^2 + k'^2$$

Letting $L = k'(\omega) = \frac{dk}{d\omega}$, then $k''(\omega) = \frac{d}{d\omega} \left(\frac{dk}{d\omega} \right) = L \frac{dL}{dk}$ and the above equation reduces to

$$2kL \frac{dL}{dk} - L^2 = \mu^2$$

or

$$\frac{2L}{L^2 + \mu^2} \frac{dL}{dk} = \frac{dk}{k}$$

which upon integration yields

$$k = \frac{l^2}{4} (\omega + m)^2 + \frac{\mu^2}{l^2} \quad (7.74)$$

where $l \neq 0$ and m are arbitrary constants.

Substituting (7.74) in (7.61), we get

$$\left[\mu^2 + \frac{l^4}{4} (\omega + m)^2 \right] \frac{\partial \omega}{\partial \psi} + \frac{l^2}{2} \rho \omega (\omega + m) = 0$$

Integrating the above equation, we obtain ω as a function of ϕ, ψ given implicitly by

$$\begin{aligned} \omega + m \ln|\omega| + \frac{4\mu^2}{l^4 m} \ln\left|\frac{\omega}{\omega + m}\right| + \frac{2\rho}{l^2} &= \chi(\phi), & m \neq 0 \\ \frac{l^2}{2} \omega - \frac{2\mu^2}{l^2 \omega} + \rho \psi &= \chi(\phi), & m = 0 \end{aligned} \quad (7.75)$$

where $\chi(\phi)$ is an arbitrary function of ϕ .

To find the flow in the physical plane, we use equations (7.48) and (7.49) with $\frac{\partial p}{\partial \phi} = 0$, $\left[\frac{d}{d\psi} \left(p + \frac{1}{2} \mu^* H_3^2 \right) - \mu^* \beta C_0 \right] = 0$, $\tilde{F} = 0$ and we have

$$\alpha(\phi, \psi) = \int \frac{\mu}{2k} \left\{ \frac{\partial \omega}{\partial \phi} d\phi + \frac{\partial \omega}{\partial \psi} d\psi \right\} \quad (7.76)$$

$$z(\phi, \psi) = \int e^{i\alpha} \sqrt{\frac{\rho}{2k}} \left\{ \frac{\mu^2 + k'^2}{\mu \rho \omega} \frac{\partial \omega}{\partial \phi} d\phi + i d\psi \right\} \quad (7.77)$$

Using (7.74) in (7.76) and integrating, we get

$$\alpha - \alpha_0 = \tan^{-1} \left(\frac{l^2}{2\mu} [\omega + m] \right); \quad \alpha_0 = \text{constant} \quad (7.78)$$

Employing (7.74) and (7.78) in (7.77) and integrating, we obtain

$$z - z_0 = \frac{l e^{i\alpha_0}}{2\mu \sqrt{2\rho}} \left\{ 2\mu \ln|\omega| + i [2\rho \psi + l^2 (\omega + m \ln|\omega|)] \right\} \quad (7.79)$$

where $z_0 = x_0 + iy_0$ is an arbitrary complex constant.

From equation (7.79), we have

$$x - x_0 = \frac{l}{2\mu \sqrt{2\rho}} \left\{ 2\mu \ln|\omega| \cos \alpha_0 - [2\rho \psi + l^2 (\omega + m \ln|\omega|)] \sin \alpha_0 \right\} \quad (7.80)$$

$$y - y_0 = \frac{l}{2\mu \sqrt{2\rho}} \left\{ 2\mu \ln|\omega| \sin \alpha_0 - [2\rho \psi + l^2 (\omega + m \ln|\omega|)] \cos \alpha_0 \right\} \quad (7.81)$$

Multiplying (7.80) by $\cos\alpha_0$, (7.81) by $\sin\alpha_0$ and adding up, we get

$$\omega = \exp \left\{ \frac{\sqrt{2\rho}}{l} [\cos\alpha_0 (x - x_0) + \sin\alpha_0 (y - y_0)] \right\} \quad (7.82)$$

Multiplying (7.80) by $\sin\alpha_0$, (7.81) by $\cos\alpha_0$ and subtracting, we have

$$2\rho\psi + l^2\omega + ml^2 \ln|\omega| = \frac{2\mu\sqrt{2\rho}}{l} [-\sin\alpha_0 (x - x_0) + \cos\alpha_0 (y - y_0)]$$

Using (7.82) in the above equation and solving for ψ , we obtain

$$\begin{aligned} \psi = \frac{\mu\sqrt{2\rho}}{\rho l} \left[\left(-\sin\alpha_0 - \frac{ml^2}{2\mu} \cos\alpha_0 \right) (x - x_0) + \left(\cos\alpha_0 - \frac{ml^2}{2\mu} \sin\alpha_0 \right) (y - y_0) \right] \\ - \frac{l^2}{2\rho} \exp \left\{ \frac{\sqrt{2\rho}}{l} [\cos\alpha_0 (x - x_0) + \sin\alpha_0 (y - y_0)] \right\} \end{aligned}$$

Having found ψ , the pressure function $p(x, y)$ is given by equations (7.2), (7.4) and (7.5), the proportionality function $\beta(x, y)$ by (7.11) and $H_3(x, y)$ by (7.8).

7.5 VORTICITY CONSTANT ON STREAMLINES.

Govindaraju [1972] investigated non-MHD viscous fluid flows when the vorticity is constant on each individual streamline. We study these confluent flows for MHD transverse-aligned case.

We assume that the streamlines and isobars do not coincide and we take $p = p(\phi)$. We further assume that $\omega = \omega(\psi)$ such that $\omega'(\psi) \neq 0$. Our requirement of $\omega'(\psi) \neq 0$ is due to the fact that $\tilde{J} = 0$ when ω is a constant and equations (7.44) to (7.47) are valid only if $\tilde{J} \neq 0$.

Employing $\omega = \omega(\psi)$ and $p = p(\phi)$ in (7.33), we obtain

$$\left. \begin{aligned} \tilde{E} &= \left(\frac{\partial k}{\partial p} + 1 \right)^2 \dot{p}^2 \\ \tilde{F} &= \left(\frac{\partial k}{\partial p} + 1 \right) \dot{p} \left(\frac{\partial k}{\partial \psi} + \rho\omega - \mu^* \beta C_0 + \frac{1}{2} \mu^* \frac{dH_3}{d\psi} \right) \\ \tilde{G} &= \mu^2 \omega'^2(\psi) + \left(\frac{\partial k}{\partial \psi} + \rho\omega - \mu^* \beta C_0 + \frac{1}{2} \mu^* \frac{dH_3}{d\psi} \right)^2 \\ \tilde{J} &= -\mu\omega'(\psi) \left(\frac{\partial k}{\partial p} + 1 \right) \dot{p} \end{aligned} \right\} \quad (7.83)$$

where prime and dot denote differentiation with respect to ψ and ϕ respectively.

Using (7.83), (7.44) yields

$$\left(\frac{\partial k}{\partial p} + 1 \right) \left[\frac{\rho\omega}{2k} - \frac{\omega''(\psi)}{\omega'(\psi)} \right] = 0$$

Since $\tilde{J} \neq 0$ and $\omega'(\psi) \neq 0$, it follows from (7.83) that $\left(\frac{\partial k}{\partial p} + 1 \right) \neq 0$ and, therefore, the above equation yields

$$\frac{\rho\omega}{2k} - \frac{\omega''(\psi)}{\omega'(\psi)} = 0 \quad (7.84)$$

From equation (7.84), we obtain that $k = k(\psi)$ so that $q = q(\psi)$ also. Employing these results and assumptions in equation (7.45), we have

$$\frac{d}{d\psi} \left[\frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)k} \right] \dot{p} = 0$$

Since $\dot{p} \neq 0$, then the above equation implies that

$$\frac{d}{d\psi} \left[\frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)k} \right] = 0$$

which upon integration gives

$$k'(\psi) + \rho\omega = B_0\mu k\omega'(\psi) \quad (7.85)$$

where B_0 is an arbitrary constant of integration.

Equations (7.84) and (7.85) form a system of two equations in two unknowns ω and k as functions of ψ . In solving this system, we consider two cases $B_0 \neq 0$ and $B_0 = 0$.

Case I. $B_0 \neq 0$: Eliminating $\rho\omega$ from (7.84) and (7.85), we get

$$\frac{k'(\psi)}{k(\psi)} + \frac{2\omega''(\psi)}{\omega'(\psi)} = B_0\mu\omega'(\psi)$$

which after one integration gives

$$\ln k + 2 \ln \omega'(\psi) = B_0\mu\omega + \ln(\rho B_1)$$

or

$$k\omega'^2(\psi) = \rho B_1 e^{B_0\mu\omega} \quad (7.86)$$

where $B_1 \neq 0$ is an arbitrary constant.

Employing equation (7.84) in (7.86) and integrating the resulting equation twice with respect to ψ , we obtain

$$-2B_1\psi - B_2\omega - \frac{\omega}{B_0^2\mu^2} e^{-B_0\mu\omega} - \frac{2}{B_0^3\mu^3} e^{-B_0\mu\omega} = B_3 \quad (7.87)$$

where B_2 and B_3 are arbitrary constants. Hence, if $B_0 \neq 0$, the function $\omega(\psi)$ is defined implicitly by equation (7.87).

To obtain the flow in the physical plane, we use equations (7.48) and (7.49). Employing $\omega = \omega(\psi)$, $p = p(\phi)$ and $k = k(\psi)$ in (7.49), we get

$$\frac{\partial \alpha}{\partial \phi} = -\frac{1}{2k} \frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)} \dot{p}$$

$$\frac{\partial \alpha}{\partial \psi} = -\frac{1}{2k} \frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)} \left(k'(\psi) + \rho\omega - \mu^* \beta C_0 + \frac{1}{2} \mu^* \frac{dH_3}{d\psi} \right)$$

Thus $\alpha(\phi, \psi)$ is given by

$$\begin{aligned} \alpha(\phi, \psi) &= \int \left(\frac{\partial \alpha}{\partial \phi} d\phi + \frac{\partial \alpha}{\partial \psi} d\psi \right) \\ &= \int -\frac{1}{2k} \frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)} \dot{p} \frac{1}{\dot{p}} dp + \int -\frac{1}{2k} \frac{k'(\psi) + \rho\omega}{\mu\omega'(\psi)} \left(k'(\psi) + \rho\omega \right. \\ &\quad \left. + \frac{1}{2} \mu^* \frac{dH_3}{d\psi} - \mu^* \beta C_0 \right) d\psi \end{aligned} \quad (7.88)$$

Using (7.85) in equation (7.88) and integrating, we get

$$\alpha(\phi, \psi) = \alpha_0 - \frac{B_0}{2} \left[p(\phi) + k(\psi) + \frac{1}{2} \mu^* H_3^2 + \gamma(\psi) \right] \quad (7.89)$$

where $\gamma'(\psi) = \rho\omega - \mu^* \beta C_0$ and α_0 is an arbitrary constant.

Employing $\omega = \omega(\psi)$, $k = k(\psi)$ and $p = p(\phi)$ in the first equation of (7.48), we find that

$$\begin{aligned} \frac{\partial z}{\partial p} &= \mp \frac{e^{i\alpha}}{\sqrt{2k}} \frac{\sqrt{\rho}}{\mu\omega'(\psi)} \\ \frac{\partial z}{\partial \psi} &= \mp e^{i\alpha} \sqrt{\frac{\rho}{2k}} \left[\frac{k'(\psi) + \rho\omega - \mu^* \beta C_0 + \frac{1}{2} \mu^* \frac{dH_3}{d\psi}}{\mu\omega'(\psi)} - i \right] \end{aligned} \quad (7.90)$$

Integrating $\frac{\partial z}{\partial p}$ with respect to p , we get

$$z = \mp \sqrt{\frac{\rho}{2k}} \frac{1}{\mu\omega'(\psi)} \left(\frac{2i}{B_0} \right) e^{i\alpha} + f(\psi) \quad (7.91)$$

where $f(\psi)$ is an arbitrary function of ψ .

Differentiating (7.91) with respect to ψ and using (7.90), we obtain

$$f'(\psi) = 0 \quad \text{or} \quad f(\psi) = z_0$$

where $z_0 = x_0 + iy_0$ is an arbitrary complex constant.

Thus, equation (7.91) becomes

$$z - z_0 = \mp \sqrt{\frac{2\rho}{k}} \frac{1}{\mu\omega'(\psi)} \frac{i}{B_0} e^{i\alpha} \quad (7.92)$$

The real and imaginary parts of (7.92) give

$$\begin{aligned} x - x_0 &= \pm \frac{1}{B_0} \sqrt{\frac{2\rho}{k}} \frac{1}{\mu\omega'(\psi)} \sin\alpha \\ y - y_0 &= \mp \frac{1}{B_0} \sqrt{\frac{2\rho}{k}} \frac{1}{\mu\omega'(\psi)} \cos\alpha \end{aligned}$$

Eliminating α from these two equations, we get

$$(x - x_0)^2 + (y - y_0)^2 = \frac{2\rho}{B_0^2 \mu^2 k \omega'^2(\psi)}$$

This equation shows that the streamlines are concentric circles with center at (x_0, y_0) and radius

$$\frac{\sqrt{2\rho}}{B_0 \mu \sqrt{k} \omega'(\psi)}$$

Case II. $B_0 = 0$: Eliminating $\rho\omega$ from (7.84) and (7.85) with $B_0 = 0$, we get

$$\frac{k'(\psi)}{k(\psi)} + \frac{2\omega''(\psi)}{\omega'(\psi)} = 0$$

Integrating with respect to ψ , we obtain

$$k\omega'^2(\psi) = \rho B_4$$

where $B_4 \neq 0$ is an arbitrary constant.

Using (7.84) in this equation and integrating twice with respect to ψ , we have

$$\omega^3 - 6B_5\omega + 12B_4\psi - 3B_6 = 0 \quad (7.93)$$

where B_5 and B_6 are arbitrary constants.

Employing $B_0 = 0$ in (6.49), we find that $\frac{\partial \alpha}{\partial \phi} = \frac{\partial \alpha}{\partial \psi} = 0$, that is

$$\alpha = \alpha_0$$

where α_0 is an arbitrary constant and the flow proceeds along parallel lines.

Hence, $\omega = \omega(\psi)$ is given by (7.87) for the vortex flow and by (7.93) for the parallel flows. Having obtained $\omega = \omega(\psi)$, the kinetic energy $k(\psi)$ can be determined by

$$k(\psi) = \frac{\rho \omega(\psi) \omega'(\psi)}{2 \omega''(\psi)}$$

Finally, using (7.84) in (7.46) and (7.47), we find that $\beta(\psi)$ and $H_3(\psi)$ are given by

$$\beta(\psi) = -C_0 \omega'(\psi) \omega_1(\psi) + \beta_0$$

and

$$H_3(\psi) = D_1 \omega(\psi) + D_2$$

where $\omega'_1(\psi) = \frac{\omega''(\psi)}{\omega(\psi) \omega'^2(\psi)}$ and β_0 , D_1 and D_2 are arbitrary constants.

Thus, summing up we have:

Theorem 7.5. *If the streamlines and isobars do not coincide and the vorticity in a two-dimensional transverse-aligned MHD flow of a viscous incompressible and electrically conducting fluid of finite electrical conductivity is constant on streamlines, then the streamlines are either parallel straight lines or concentric circles.*

CHAPTER 8

VISCOELASTIC FLUID FLOW IMPINGING ON A WALL WITH SUCTION OR BLOWING

8.1 INTRODUCTION.

During the past decade, there has been substantial interest in flows of viscoelastic liquids due to the occurrence of these liquids in industrial processes such as polymers in polymer technology or liquid metal in metallurgy. Behaviour of viscoelastic liquids cannot be accurately described by the Newtonian fluid model. The process of suction and blowing has its importance in dealing with the manufacturing of polymers, ceramics, paints and other second-grade or viscoelastic fluids. Suction is applied to chemical processes to remove reactants. Blowing is used to add reactants, cool the surfaces and prevent corrosion. The boundary value problem associated with viscoelastic second-grade fluid flow is of third order and, therefore, to determine the solution one requires an additional condition when compared with the boundary value problem for Newtonian fluids. One of the methods that successfully overcomes this requirement of an additional condition is the perturbation technique.

One class of flows which has received considerable attention is stagnation point flow. The solution for stagnation point flow of Newtonian fluid was first proposed by Blasius [1908] and the resulting differential equation was first integrated numerically

by Hiemenz [1911]. The numerical work was later improved by Howarth [1935]. In a stagnation point flow of a Newtonian fluid, a rigid wall occupies the entire x -axis, the fluid domain is $y > 0$ and the flow impinges on the wall either orthogonally (c.f. Hiemenz [1911], Goldstein [1965]) or obliquely (c.f. Stuart [1954], Dorrepaal [1986]). In a study of Newtonian fluid obliquely impinging on a flat rigid wall, Dorrepaal [1986] found that the slope of the dividing streamline at the wall divided by its slope at infinity is independent of the angle of incidence.

Beard and Walters [1964] used boundary-layer equations to study two-dimensional flow near a stagnation point of a viscoelastic fluid when fluid is impinging on the wall orthogonally. Peddieson [1971] employed the finite difference method in combination with the perturbation method and the concept of local similarity to solve flow problems such as flow near a stagnation point, flow past a flat plate and flow with exponential pressure gradient.

In a recent paper, Dorrepaal, Chandna and Labropulu [1992] investigated the behaviour of a viscoelastic second-grade fluid impinging a flat rigid wall at an arbitrary angle of incidence and showed that the ratio of the two slopes depends on the elastic effects of the fluid. Labropulu, Dorrepaal and Chandna [1992] investigated the behaviour of a second-grade fluid impinging on a porous wall at a arbitrary angle of incidence with suction or blowing on the wall.

In this chapter, we study the behaviour of a second-grade fluid impinging on a porous wall at an arbitrary angle of incidence with suction or blowing on the wall. In particular, we investigate the behaviour of the fluid for various magnitudes of suction or blowing and the effects of elasticity of the fluid. The choice of the second-grade fluid model lies in the consideration that this fluid model describes quite well the behaviour of liquids with short memories, and hence, it suits the assumption of small elasticity. As reported by Markovitz and Coleman [1964], in low shear rate

flow the second-grade model can be used to describe the behaviour of some dilute polymer solutions such as the 5.4 percent solution of polyisobutylene in cetane.

8.2. FLOW EQUATIONS.

The steady plane flow of a viscous incompressible second-grade fluid is governed by equations (2.19) to (2.21). These equations consist of a system of three equations in three unknowns, the unknowns being the velocity components u , v and the fluid pressure p .

We non-dimensionalize equations (2.19) to (2.21) according to

$$\begin{aligned} x^* &= \sqrt{\frac{\beta}{\nu}} x, & y^* &= \sqrt{\frac{\beta}{\nu}} y \\ u^* &= \frac{1}{\sqrt{\nu\beta}} u, & v^* &= \frac{1}{\sqrt{\nu\beta}} v, & p^* &= \frac{1}{\rho\nu\beta} p \end{aligned}$$

where β has units of inverse time and $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity. The flow equations in non-dimensional form after dropping the star are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (8.1)$$

$$\begin{aligned} u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \frac{\partial p}{\partial x} &= \nabla^2 u - We \left\{ \frac{\partial}{\partial x} \left[2u \frac{\partial^2 u}{\partial x^2} + 2v \frac{\partial^2 u}{\partial x \partial y} + 4 \left(\frac{\partial u}{\partial x} \right)^2 \right. \right. \\ &+ 2 \frac{\partial v}{\partial x} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \left. \right] + \frac{\partial}{\partial y} \left[\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \right] \right\} \\ &+ \frac{\alpha_2 \beta}{\mu} \left\{ \frac{\partial}{\partial x} \left[4 \left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} \end{aligned} \quad (8.2)$$

$$\begin{aligned} u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \frac{\partial p}{\partial y} &= \nabla^2 v - We \left\{ \frac{\partial}{\partial x} \left[\left(u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} \right) \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) + 2 \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} \right. \right. \\ &+ 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} \left. \right] + \frac{\partial}{\partial y} \left[2u \frac{\partial^2 v}{\partial x \partial y} + 2v \frac{\partial^2 v}{\partial x^2} + 4 \left(\frac{\partial v}{\partial y} \right)^2 + 2 \frac{\partial u}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right] \right\} \\ &+ \frac{\alpha_2 \beta}{\mu} \left\{ \frac{\partial}{\partial y} \left[4 \left(\frac{\partial v}{\partial y} \right)^2 + \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right)^2 \right] \right\} \end{aligned} \quad (8.3)$$

where $We = -\frac{\alpha_1\beta}{\mu}$, the Weissenberg number, is the ratio of elastic effects to viscous effects.

Continuity equation (8.1) implies the existence of a streamfunction $\psi(x, y)$ such that

$$u = \frac{\partial\psi}{\partial y}, \quad v = -\frac{\partial\psi}{\partial x} \quad (8.4)$$

Substitution of (8.4) in equations (8.2) and (8.3) and elimination of pressure from the resulting equations using $\frac{\partial^2 p}{\partial x \partial y} = \frac{\partial^2 p}{\partial y \partial x}$ yields

$$\frac{\partial(\psi, \nabla^2 \psi)}{\partial(x, y)} + \nabla^4 \psi + We \frac{\partial(\psi, \nabla^4 \psi)}{\partial(x, y)} = 0 \quad (8.5)$$

Having obtained a solution of equation (8.5), the velocity components are given by (8.4) and the pressure can be found by integrating equations (8.2) and (8.3).

Substituting (2.5) and (2.6) in (2.4), non-dimensionalizing and using (8.4) in the resulting equation, the shear stress component τ_{12} of \underline{T} is given by

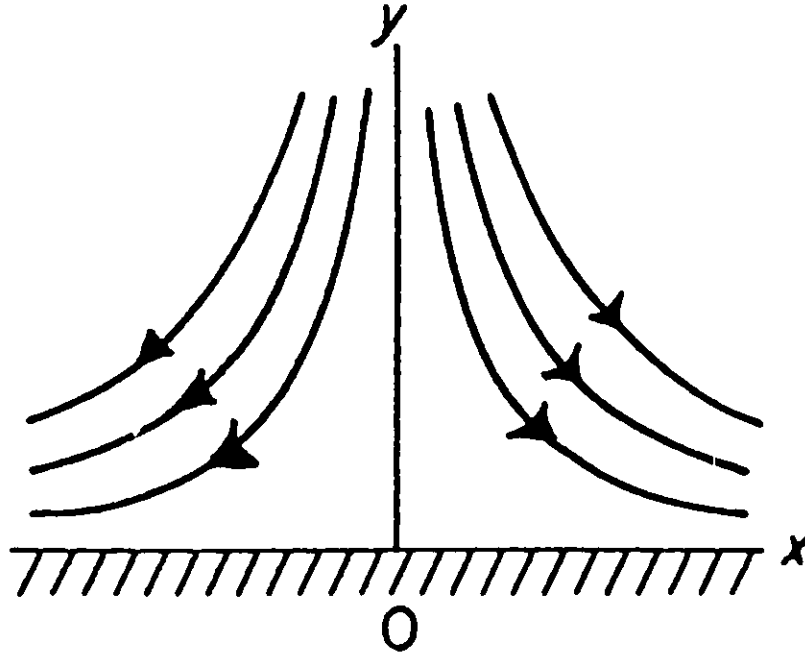
$$\begin{aligned} \tau_{12} = \mu\beta \left\{ \frac{\partial^2 \psi}{\partial y^2} - \frac{\partial^2 \psi}{\partial x^2} - We \left[\frac{\partial \psi}{\partial y} \left(\frac{\partial^3 \psi}{\partial x \partial y^2} - \frac{\partial^3 \psi}{\partial x^3} \right) - \frac{\partial \psi}{\partial x} \left(\frac{\partial^3 \psi}{\partial y^3} - \frac{\partial^3 \psi}{\partial x^2 \partial y} \right) \right. \right. \\ \left. \left. + 2 \frac{\partial^2 \psi}{\partial x \partial y} \frac{\partial^2 \psi}{\partial y^2} + 2 \frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial x \partial y} \right] \right\} \end{aligned} \quad (8.6)$$

8.3 ORTHOGONAL FLOW.

Consider a hyperbolic flow impinging on a flat wall at $y = 0$ with constant suction or blowing. The non-dimensional boundary conditions are given by

$$\left. \begin{aligned} \frac{\partial \psi}{\partial y} = 0, \quad \frac{\partial \psi}{\partial x} = \pm \frac{v_0}{\sqrt{\nu \beta}} \quad \text{at } y = 0 \\ \psi(x, y) \sim xy, \quad \text{as } y \rightarrow \infty \end{aligned} \right\} \quad (8.7)$$

where v_0 is the constant suction or blowing velocity in the y -direction. The positive sign denotes suction and the negative sign denotes blowing.



Two dimensional orthogonal flow

Following Blasius [1908], we assume that

$$\psi(x, y) = x f(y) \quad (8.8)$$

Using equation (8.8) in (8.5) and (8.7) we find that, after one integration, $f(y)$ must satisfy

$$f''' + ff'' - f'^2 + 1 + We \left[ff^{(iv)} - 2f'f''' + f''^2 \right] = 0 \quad (8.9)$$

$$\left. \begin{aligned} f(0) = s, \quad f'(0) = 0 \\ f'(\infty) = 1 \end{aligned} \right\} \quad (8.10)$$

where prime denotes differentiation with respect to y and $s = \pm \frac{v_0}{\sqrt{\nu\beta}}$.

Due to the presence of elasticity in the fluid, we note that the differential equation (3.9) is of fourth order, whereas in the ordinary viscous case ($We = 0$) the equation is of order three. Thus, it would appear that an additional boundary condition must be imposed to obtain a solution. Previous experience with the second-grade model (c.f. Craik [1968]) indicates that it is valid only for slightly viscoelastic liquids. In this interpretation the second grade constitutive equation is seen as resulting from a two-term asymptotic expansion of a very general constitutive equation for small elastic effects. To be consistent with the order of accuracy inherent in the second-grade constitutive equation it is clear that the solution of (8.9) should also have the form of a two-term asymptotic expansion for small elastic effects. Thus, following Beard and Walters [1964], we assume that

$$f(y) = f_0(y) + We f_1(y) + O(We^2) \quad (8.11)$$

Employing (8.11) in (8.9), we obtain

$$\begin{aligned} f_0''' + f_0 f_0'' - f_0'^2 + 1 + We \left[f_1''' + f_0 f_1'' - 2f_0' f_1' + f_0'' f_1 + f_0 f_0^{(iv)} \right. \\ \left. - 2f_0' f_0''' + f_0''^2 \right] + O(We^2) = 0 \end{aligned} \quad (8.12)$$

The terms of $O(1)$ yield an ordinary differential equation for $f_0(y)$ and the terms of $O(We)$ yield an equation for $f_1(y)$.

The boundary value problem for $f_0(y)$ is

$$\left. \begin{aligned} f_0''' + f_0 f_0'' - f_0'^2 + 1 &= 0 \\ f_0(0) = s, \quad f_0'(0) &= 0 \\ f_0'(\infty) &= 1 \end{aligned} \right\} \quad (8.13a,b,c)$$

When $s = 0$, system (8.13) defines the well-known Hiemenz function (c.f. Hiemenz [1911]) describing the stagnation point flow of a Newtonian fluid.

We solve system (8.13) numerically using a shooting method (c.f. Johnson and Riess [1982]) for different values of s . Equation (8.13a) can be written as a system of first order differential equations as follows

$$\begin{bmatrix} f_0'(y) \\ g_0'(y) \\ m_0'(y) \end{bmatrix} = \begin{bmatrix} g_0(y) \\ m_0(y) \\ g_0^2(y) - f_0(y) m_0(y) - 1 \end{bmatrix} \quad (8.14)$$

where $f_0'(y) = g_0(y)$ and $f_0''(y) = g_0'(y) = m_0(y)$.

The boundary conditions (8.13b) can be written as

$$\begin{bmatrix} f_0(0) \\ g_0(0) \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} \quad (8.15)$$

The boundary condition (8.13c) is replaced by

$$f_0''(0) = g'(0) = m_0(0) = C \quad (8.16)$$

System (8.14) with boundary conditions (8.15) and (8.16) are then solved numerically. However, in solving this system we must choose C such that the solution to problem (8.14) to (8.16) also satisfies the infinity condition (8.13c). Starting with an initial value of C , we employ the fourth order Runge-Kutta method to integrate (8.14) to obtain f_0' and f_0'' at a sufficiently large distance from the wall located at $y = 0$. If the value of $\gamma = f_0' - 1$ is positive, we choose another value of C for which $\gamma = f_0' - 1$ is negative or vice-versa. Then the interval between these two values

of C is bisected repeatedly to obtain a new C and the above Runge-Kutta scheme used to integrate system (8.14) for each of these new values of C until $\gamma = f'_0 - 1$ approaches 0. This numerical procedure is stopped when the difference in the value of $\gamma = f'_0 - 1$ far away from the wall for two successive values of C is less than 10^{-8} .

Having obtained f_0 , f'_0 and f''_0 numerically, f'''_0 , $f_0^{(iv)}$ and $f_0^{(v)}$ can be determined by

$$\begin{aligned} f'''_0 &= f_0'^2 - f_0 f''_0 - 1 \\ f_0^{(iv)} &= f'_0 f''_0 - f_0 f'''_0 \\ f_0^{(v)} &= f_0''^2 - f_0 f_0^{(iv)} \end{aligned} \quad (8.17)$$

From the numerical solution we have, for small y

$$f_0(y) = s + \frac{1}{2}Cy^2 - \frac{1}{6}(1 + sC)y^3 + O(y^4) \quad (8.18)$$

and for large y

$$f_0(y) \sim y + A \quad (8.19)$$

where $C = f''_0(0)$ and A are given in Table 1 for different values of s .

The boundary value problem for $f_1(y)$ is given by

$$\left. \begin{aligned} f_1''' + f_0 f_1'' - 2f'_0 f_1' + f''_0 f_1 &= 2f'_0 f_0''' - f_0 f_0^{(iv)} - f_0''^2 \\ f_1(0) = f_1'(0) &= 0 \\ f_1'(\infty) &= 0 \end{aligned} \right\} \quad (8.20)$$

Letting

$$f_1(y) = M(y) - f''_0(y) \quad (8.21)$$

in system (8.20), we obtain

$$\left. \begin{aligned} M''' + f_0 M'' - 2f'_0 M' + f''_0 M &= f_0^{(v)} \\ M(0) = C, \quad M'(0) &= -1 - sC \\ M'(\infty) &= 0 \end{aligned} \right\} \quad (8.22)$$

Using the numerical solution obtained above for f_0' , f_0'' and $f_0^{(v)}$, we integrate system (8.22) numerical using the shooting method as outlined above.

The numerical integration of system (8.22) gives

$$M''(0) = D, \quad M(\infty) = A_1 \quad (8.23)$$

where D and A_1 are given in Table 1 for different values of s . When $s = 0$, the values of C , A , D and A_1 are in good agreement with those of Beard and Walters [1964].

The solution of equation (8.9) is, therefore, given by

$$f = f_0 + We(M - f_0'') + O(We^2)$$

The function $f(y; We)$ has the form, for small y ,

$$f(y; We) = s + \frac{1}{2} [C + We(D - s - s^2 C)] y^2 - \frac{1}{6} [1 + sC + We(sD + C^2)] y^3 + O(y^4) \quad (8.24)$$

and for large y ,

$$f(y) \sim y + (A + WeA_1) \quad (8.25)$$

s	C	A	D	A_1
0	1.232588	-0.647900	1.139019	0.825547
0.1	1.290975	-0.524737	1.400554	0.862231
1.0	1.889314	0.540678	7.151415	1.333555
2.0	2.570056	1.657803	26.746598	2.087723
3.0	3.526640	2.632895	70.772995	2.976356
-0.1	1.176041	-0.772168	0.924185	0.792135
-1.0	0.756575	-1.944982	0.190120	0.618700
-2.0	0.475811	-3.361625	0.081132	0.541119
-3.0	0.329453	-4.858121	0.033368	0.445974

Table 8.1. Numerical values of C , A , D and A_1 .

Employing (8.8) and (8.11) in (8.6), we have

$$\tau_{12} = \mu\beta x \{f_0'' + We(f_1'' - 3f_0'f_0'' + f_0f_0''')\} + O(We^2) \quad (8.26)$$

Let τ_w be the shear stress at the wall $y = 0$. Applying the boundary conditions for $f_0(y)$ and $f_1(y)$ in (8.26), we have

$$\tau_w = \tau_{12}(x, 0) = \mu\beta x \{f_0''(0) + We[f_1''(0) + s f_0'''(0)]\}$$

Thus, the shear stress on $y = 0$ is given by

$$\tau = \frac{1}{\mu\beta x} \tau_w = C + We[D - 2s(1 + sC)]$$

where C and D are given in Table 1 for different values of s . Numerical values of τ for different values of s and We are given in Table 2. The numerical values for τ are in good agreement with those obtained by Beard and Walters [1964] for $s = 0$ and those obtained by Nguyen [1991] for $s > 0$.

s	0.0	1.0	2.0	3.0	-1.0	-2.0
We						
0.00	1.232588	1.889314	2.670056	3.526640	0.756575	0.475811
0.01	1.243978	1.903042	2.683918	3.539575	0.763344	0.478557
0.05	1.289539	1.957953	2.739363	3.591314	0.790423	0.489543
0.075	1.318014	1.992273	2.774017	3.623650	0.807347	0.49640
0.1	1.346490	2.026593	2.808671	3.655987	0.824272	0.503275
0.5	1.802097	2.575707	3.363131	4.173377	1.09506	0.613133

Table 8.2. Numerical values of τ .

From Table 2, we see that as We increases, the shear stress on the wall increases. Similarly, as s increases, the shear on the wall increases as well. Thus, the effects of elasticity and the effects of suction or blowing are to increase the shear stress on the wall.

Figure 8.1 shows the profiles of f' for various We numbers when $s=1$. We observed that as the elasticity of the fluid increases, the velocity near the wall increases. The same effect is observed for other values of s . Figure 8.2 shows

various f' for a fixed We number, $We = 0.1$, and different magnitudes of suction or blowing on the wall. It can be seen that as the value of s increases, the velocity near the wall increases. The ratio of the velocity near the wall to that at infinity is given by $f'(y)$. From Figure 8.1 and 8.2, it is observed that the velocity near the wall exceeds that at infinity for this viscoelastic fluid. Therefore, the effects of elasticity and suction are to increase the velocity of the fluid near the wall.

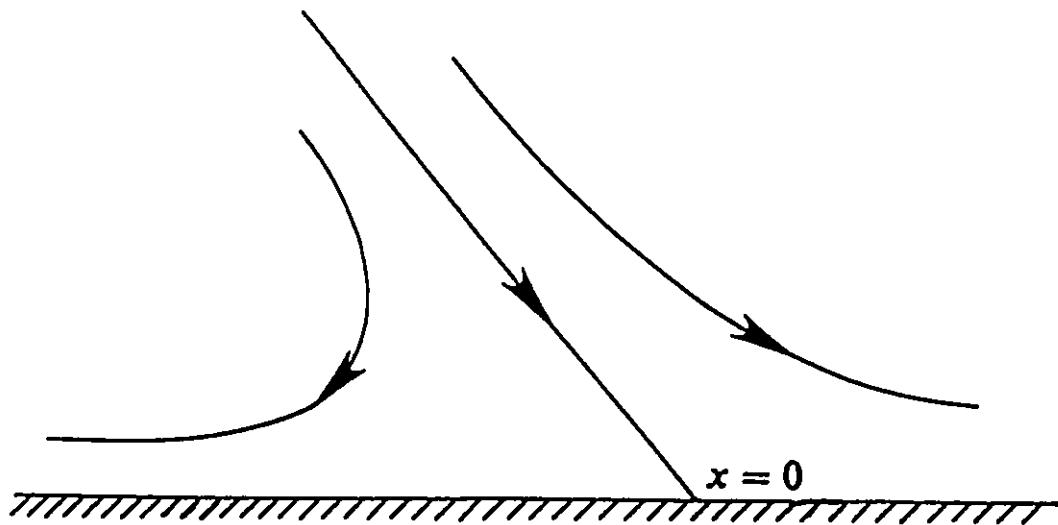
8.4 OBLIQUE FLOW.

Following Stuart [1959], we assume that the streamfunction far from the wall is given by

$$\psi(x, y) = ky^2 + xy + \alpha \quad (8.27)$$

where k is a constant, $\alpha = 0$ when $s \geq 0$, $\alpha = G(y_0)$ when $s < 0$ and y_0 will be numerically determined later. The dividing streamline which comes into the wall from infinity is defined by $\psi(x, y) = \alpha$ and its slope at infinity is $-\frac{1}{k}$. Equation (8.27) suggests $\psi(x, y)$ has the form

$$\psi(x, y) = xF(y) + G(y) \quad (8.28)$$



Two dimensional oblique flow

The boundary conditions that $F(y)$ and $G(y)$ satisfy are

$$\left. \begin{aligned} F(0) = s, \quad F'(0) = 0, \quad G(0) = G'(0) = 0 \\ F(y) \sim y, \quad G(y) \sim ky^2 + \alpha, \quad \text{as } y \rightarrow \infty \end{aligned} \right\} \quad (8.29)$$

Employing equation (8.29) in (8.5), we obtain an equation which contains terms of $O(x)$ and $O(1)$. The terms of $O(x)$ yield an ordinary differential equation for $F(y)$ while the terms of $O(1)$ yield an equation for $G(y)$.

The boundary value problem for $F(y)$, after one integration, is

$$\left. \begin{aligned} F''' + FF'' - F'^2 + 1 + We \{ FF^{(iv)} - 2F'F''' + F''^2 \} = 0 \\ F(0) = s, \quad F'(0) = 0, \quad F'(\infty) = 1 \end{aligned} \right\} \quad (8.30)$$

Solutions of system (8.30) are identical to those of system (8.9) and (8.10). Therefore, from equations (8.24) and (8.25) it follows that, for small y

$$F(y; We) = s + \frac{1}{2} [C + We(D - s - s^2C)] y^2 - \frac{1}{6} [1 + sC + We(sD + C^2)] y^3 + O(y^4) \quad (8.31)$$

and for large y

$$F(y) \sim y + (A + WeA_1) \quad (8.32)$$

where C, D, A and A_1 are given in Table 1.

The boundary-value problem for $G(y)$ is given by

$$\left. \begin{aligned} G^{(iv)} + FG''' - F'G'' + We \{ FG^{(iv)} - F^{(iv)}G' \} = 0 \\ G(0) = G'(0) = 0, \quad G''(\infty) = 2k \end{aligned} \right\} \quad (8.33)$$

Integrating equation (8.33) once and using the asymptotic behaviours of $F(y)$ and $G(y)$ for large y to determine the constant of integration, we obtain

$$G''' + FG'' - F'G' + We \{ FG^{(iv)} - F'G''' + F''G'' - F'''G' \} = 2k(A + WeA_1) \quad (8.34)$$

Making a substitution of the form

$$G'(y) = 2kH(y) \quad (8.35)$$

equation (8.34) takes the form

$$H'' + FH' - F'H + We \{FH''' - F'H'' + F''H' - F'''H\} = A + WeA_1 \quad (8.36)$$

The boundary conditions for $H(y)$ are

$$H(0) = 0, \quad H'(\infty) = 1 \quad (8.37)$$

We assume a solution for $H(y)$ of the form

$$H(y) = H_0(y) + WeH_1(y) + O(We^2) \quad (8.38)$$

The boundary value problem for $H_0(y)$ is given by

$$\left. \begin{aligned} H_0'' + F_0H_0' - F_0'H_0 &= A \\ H_0(0) = 0, \quad H_0'(\infty) &= 1 \end{aligned} \right\} \quad (8.39)$$

This system is solved numerically using the above shooting method. System (8.39) has been solved by Dorrepaal [1986] for $s = 0$. From the numerical solutions of this system we have

$$H_0(y) = Ey + \frac{1}{2}y^2[A - sE] + O(y^3) \quad (8.40)$$

where E is given in Table 3 for different values of s .

s	E
0	1.406544
0.1	1.391607
1.0	1.270384
2.0	1.176233
3.0	1.470608
-0.1	1.421555
-1.0	1.543394
-2.0	1.601674
-3.0	1.600534

Table 8.3. Numerical values of E .

The profiles of H'_0 for different values of s are shown in Figure 8.3. The boundary-value problem for $H_1(y)$ is given by

$$\left. \begin{aligned} H_1'' + F_0 H_1' - F_0' H_1 + F_0 H_0''' - F_0' H_0'' + F_0'' H_0' - F_0''' H_0 + F_1 H_0' - F_1' H_0 &= A_1 \\ H_1(0) = 0, \quad H_1'(\infty) &= 0 \end{aligned} \right\} \quad (8.41)$$

Setting

$$H_1(y) = h(y) - H_0''(y) \quad (8.42)$$

system (8.41) becomes

$$\left. \begin{aligned} h'' + F_0 h' - F_0' h &= L(y) \\ h(0) = A - sE, \quad h'(\infty) &= 0 \end{aligned} \right\} \quad (8.43)$$

where $L(y) = A_1 + H_0^{(\text{iv})} + H_0 M' - H_0' M$.

The function $F_0''(y)$ is a solution of the homogeneous equation in (8.43). Reduction of order can be used to generate the second solution of this equation and thus we have

$$h(y) = \frac{A - sE}{C} F_0''(y) + F_0''(y) \int_0^y \frac{K(z)}{F_0''(z)I(z)} dz \quad (8.44)$$

where

$$\begin{aligned} K(z) &= \int_0^z L(\tau) I(\tau) d\tau \\ I(\tau) &= F_0''(\tau) \exp \left[\int_0^\tau F_0(s) ds \right] \end{aligned}$$

For small y , we have

$$h(y) = A - sE - \frac{A - sE}{C} (1 + sC)y + \frac{1}{2} \left[-\frac{A - sE}{C} s + A_1 \right] y^2 + O(y^3) \quad (8.45)$$

Using (8.45) in (8.42) we find for small y

$$H_1(y) = \frac{sE - A}{C} y + \frac{1}{2} \left[\frac{A - sE}{C} (s + s^2 C) + A_1 - EC \right] y^2 + O(y^3) \quad (8.46)$$

Employing (8.38), (8.40) and (8.46) in (8.35), after one integration, we get

$$G(y) = \left(E - \frac{A - sE}{C} We\right) ky^2 + \frac{k}{3} \left\{ (A - sE) + We \left[\frac{A - sE}{C} (s + s^2 C) + A_1 - EC \right] \right\} y^3 + O(y^4) \quad (8.47)$$

Substituting the expressions for $F(y)$ and $G(y)$ given by (8.31) and (8.47) in equation (8.28), we find that the streamfunction $\psi(x, y)$ for small y has the form

$$\begin{aligned} \psi(x, y) = sx + & \left\{ \frac{x}{2} [C + We(D - s - s^2 c)] + \left(E - \frac{A - sE}{C} We\right) k \right\} y^2 \\ & - \left\{ \frac{x}{6} [1 + sC + We(sD + C^2)] - \frac{k}{3} [A - sE \right. \\ & \left. + We \left(\frac{A - sE}{C} (s + s^2 c) + A_1 - EC \right)] \right\} y^3 + O(y^4) \end{aligned} \quad (8.48)$$

Using equation (8.48) in (8.6) and evaluating the resulting equation at $y = 0$, we find that the shear stress on the wall $y = 0$ is given by

$$\begin{aligned} \tau_w = \tau_{12}(x, 0) = \mu\beta x \{ & C + We[D - 2s(1 + sC)] \} \\ & + 2k\mu\beta \left\{ E + We(A - sE) \left[s - \frac{1}{C} \right] \right\} + O(We^2) \end{aligned} \quad (8.49)$$

8.5 DISCUSSION.

The orthogonal and oblique flow of a non-Newtonian second-grade fluid impinging on a porous flat wall are investigated when suction or blowing is applied on the wall. The porous wall is at $y = 0$ and the fluid flow occupies the entire upper half plane.

In the case of suction, that is when the values of s are positive, the fluid penetrates the wall and the slope of the streamlines on the wall is infinite. The streamline pattern for the orthogonal and oblique flows when suction is applied are shown in Figures 8.4 and 8.5 respectively. For graphing purposes we took $k = 1$, $s = 1$ and $We = 0.0, 0.2$ and 0.5 .

Considering the case when blowing is applied, that is when s is negative, we find that there exists a $y_0 > 0$ depending on s and We such that $F(y_0) = F_0(y_0) + WeF_1(y_0) = 0$. Numerical values of y_0 are given in Table 4 for different values of s and We .

The line $y = y_0$ behaves as an imaginary wall and fluid flows on both sides of this wall. Thus, the flow near $y = y_0$ needs to be analyzed further. The dividing streamline is given by

$$\psi(x, y) = xF(y) + G(y) = G(y_0)$$

Using Taylor's series expansion for $F(y)$ and $G(y)$ about y_0 and employing $F(y_0) = 0$, we have

$$(y - y_0) \left\{ [xF'(y_0) + G'(y_0)] + \frac{1}{2}[xF''(y_0) + G''(y_0)](y - y_0) + \frac{1}{6}[xF'''(y_0) + G'''(y_0)](y - y_0)^2 + \text{H.O.T.} \right\} = 0$$

The streamline $\psi(x, y) = G(y_0)$ consists of two parts

$$y = y_0$$

and

$$\begin{aligned} [xF'(y_0) + G'(y_0)] + \frac{1}{2}[xF''(y_0) + G''(y_0)](y - y_0) \\ + \frac{1}{6}[xF'''(y_0) + G'''(y_0)](y - y_0)^2 + \text{H.O.T.} = 0 \end{aligned} \quad (8.50)$$

We should consider two cases: (1) $F'(y_0) \neq 0$ and (2) $F'(y_0) = 0$.

Case 1 : $F'(y_0) \neq 0$

Dividing (8.50) by $F'(y_0)$ and defining a new variable X given by

$$X = x + \frac{G'(y_0)}{F'(y_0)},$$

equation (8.50) takes the form

$$\begin{aligned} X + \frac{1}{2} \left[\frac{G''(y_0)}{F'(y_0)} - \frac{G'(y_0)}{F'(y_0)} \frac{F''(y_0)}{F'(y_0)} \right] (y - y_0) \\ + \frac{1}{2} \frac{F''(y_0)}{F'(y_0)} X(y - y_0) + \text{H.O.T.} = 0 \end{aligned} \quad (8.51)$$

From equation (8.51), we see that the dividing streamline meets the line $y = y_0$ at $X = 0$ or at $x = -\frac{G'(y_0)}{F'(y_0)}$ and its slope m_s near $y = y_0$ is

$$m_s = \frac{2F'^2(y_0)}{G'(y_0)F''(y_0) - G''(y_0)F'(y_0)} \quad (8.52)$$

Employing equation (8.36) in (8.52), we have

$$m_s = \frac{F'^2(y_0)}{k[H(y_0)F''(y_0) - H'(y_0)F'(y_0)]}$$

Letting m to be the slope of the dividing streamline far from the wall, we find that $m = -\frac{1}{k}$. Thus, the ratio $R = \frac{m_s}{m}$ is found to be

$$R = \frac{m_s}{m} = \frac{F'^2(y_0)}{H'(y_0)F'(y_0) - H(y_0)F''(y_0)} \quad (8.53)$$

Using $F(y) = F_0(y) + WeF_1(y)$ and $H(y) = H_0(y) + WeH_1(y)$, and neglecting terms of $O(We^2)$, equation (8.48) becomes

$$\begin{aligned} R = \frac{F_0'^2(y_0)}{H_0'(y_0)F_0'(y_0) - H_0(y_0)F_0''(y_0)} + We \left\{ \frac{2F_0'(y_0)F_1'(y_0)}{H_0'(y_0)F_0'(y_0) - H_0(y_0)F_0''(y_0)} \right. \\ \left. - \frac{F_0'^2[F_0'(y_0)H_1'(y_0) + H_0'(y_0)F_1'(y_0) - F_0''(y_0)H_1(y_0) - H_0(y_0)F_1''(y_0)]}{[H_0'(y_0)F_0'(y_0) - H_0(y_0)F_0''(y_0)]^2} \right\} \end{aligned} \quad (8.54)$$

Numerical values of R are given in Table 5 for different values of We and s .

Case 2 : $F'(y_0) = 0$

In this case $y_0 = s = 0$ and $G'(y_0) = 0$. Hence equation (8.50) gives

$$\frac{1}{2}[xF'''(0) + G'''(0)]y + \frac{1}{6}[xF''''(0) + G''''(0)]y^2 + \text{H.O.T.} = 0$$

Dividing this equation by $F'''(0) \neq 0$ and letting $X = x + \frac{G'''(0)}{F'''(0)}$, we have

$$\frac{1}{2}y \left\{ X + \frac{1}{3} \left[\frac{G''''(0)}{F'''(0)} - \frac{G''(0)F''''(0)}{F''^2(0)} \right] y + \text{H.O.T.} \right\} = 0 \quad (8.55)$$

From (8.55), we can see that the dividing streamline meets the wall $y = 0$ at $X = 0$ or $x = -\frac{G'''(0)}{F'''(0)}$ and its slope m_s near $y = 0$ is

$$m_s = -3 \frac{F''^2(0)}{G''''(0)F'''(0) - G''(0)F''''(0)} \quad (8.56)$$

The ratio $R = \frac{m_s}{m}$, where m is the slope of the dividing streamline far from the wall, is found to be

$$R = 3k \frac{F''^2(0)}{G''''(0)F'''(0) - G''(0)F''''(0)} \quad (8.57)$$

Employing (8.31) and (8.47) in (8.57) and neglecting terms of order $O(We^2)$, we get

$$\begin{aligned} R &= \frac{3}{2} \frac{C^2}{AC + E} \left\{ 1 + We \left[\frac{D}{C} + \frac{DE - A_1 C^2 + A}{C(AC + E)} \right] \right\} \\ &= 3.748513 [1 + We(0.523675)] \end{aligned} \quad (8.58)$$

From equations (8.54) and (8.58), we can conclude that R is independent of k but dependent on the Weissenberg number We . Thus, the slope of the dividing streamline at $y = y_0$ divided by its slope at infinity is the same for all oblique flows for the second-grade fluid that we have considered. The same results are also true for the Newtonian fluid (c.f. Dorreppal [1986]). The streamline pattern for orthogonal and oblique flows with blowing are shown in Figures 8.6 and 8.7 respectively when $k = 1$, $s = -1$ and $We = 0.2$.

s We	0.0	-0.1	-1.0	-2.0	-3.0
0.0	0	0.436545	1.879491	3.337490	4.845499
0.05	0	0.427499	1.858499	3.314499	4.825499
0.075	0	0.423499	1.847499	3.302499	4.815499
0.1	0	0.419499	1.837499	3.291499	4.805499
0.2	0	0.403499	1.797499	3.247499	4.766499
0.5	0	0.365499	1.687499	3.119499	4.650499

Table 8.4. Numerical values of y_0 .

s We	0.0	-0.1	-1.0	-2.0	-3.0
0.0	3.748513	3.706229	2.352178	2.278485	1.994594
0.05	3.846663	3.791902	2.350447	2.107440	1.984851
0.075	3.895738	3.838691	2.347987	2.106337	1.979755
0.1	3.944813	3.886158	2.345410	2.103372	1.974521
0.2	4.141114	4.082899	2.334955	2.089606	1.950463
0.5	4.730014	4.690004	2.276687	2.021446	1.861978

Table 8.5. Numerical values of $R = \frac{m_s}{m}$.

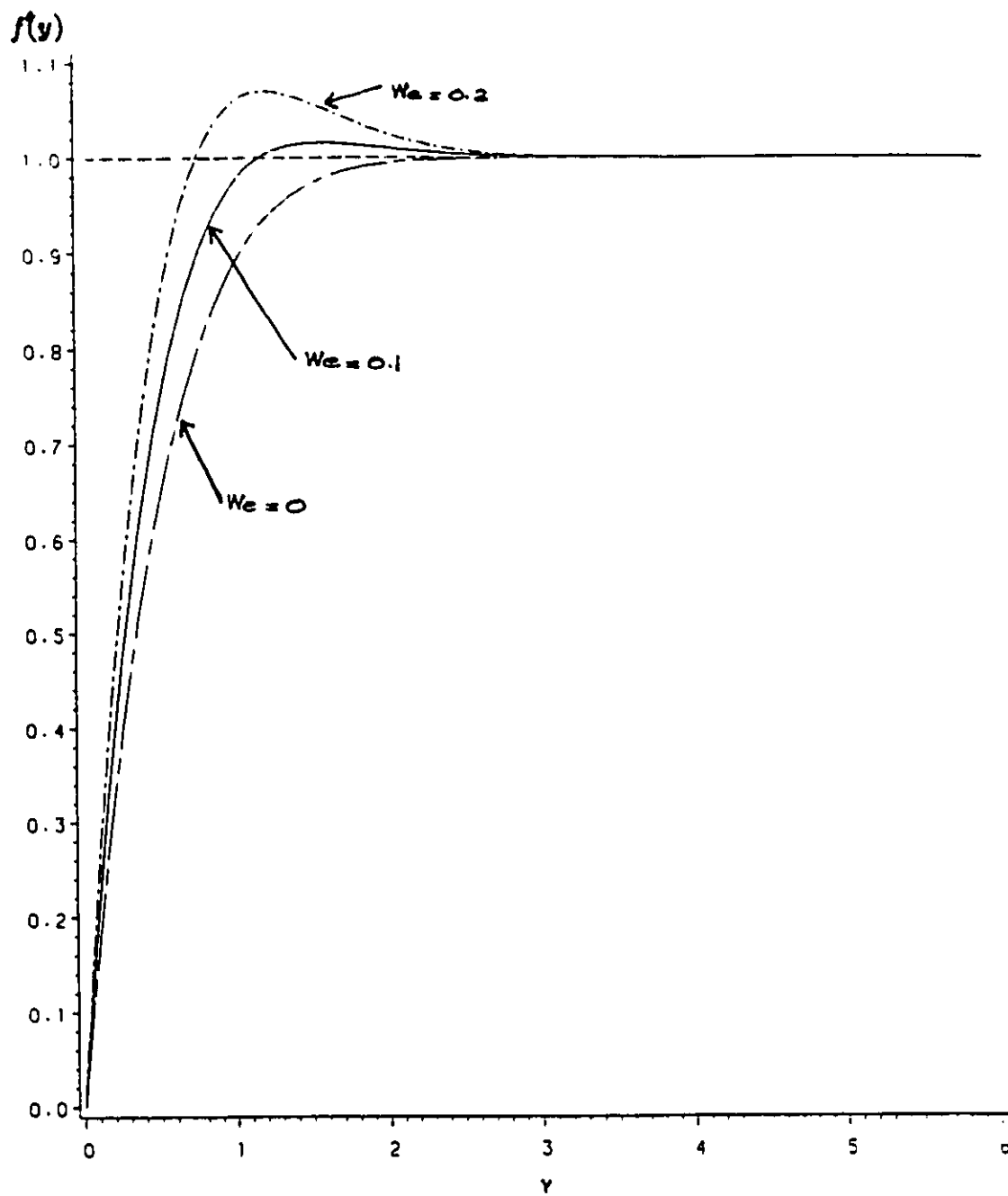


FIGURE 8.1: Velocity profiles f' for various We numbers when $s = 1.0$.

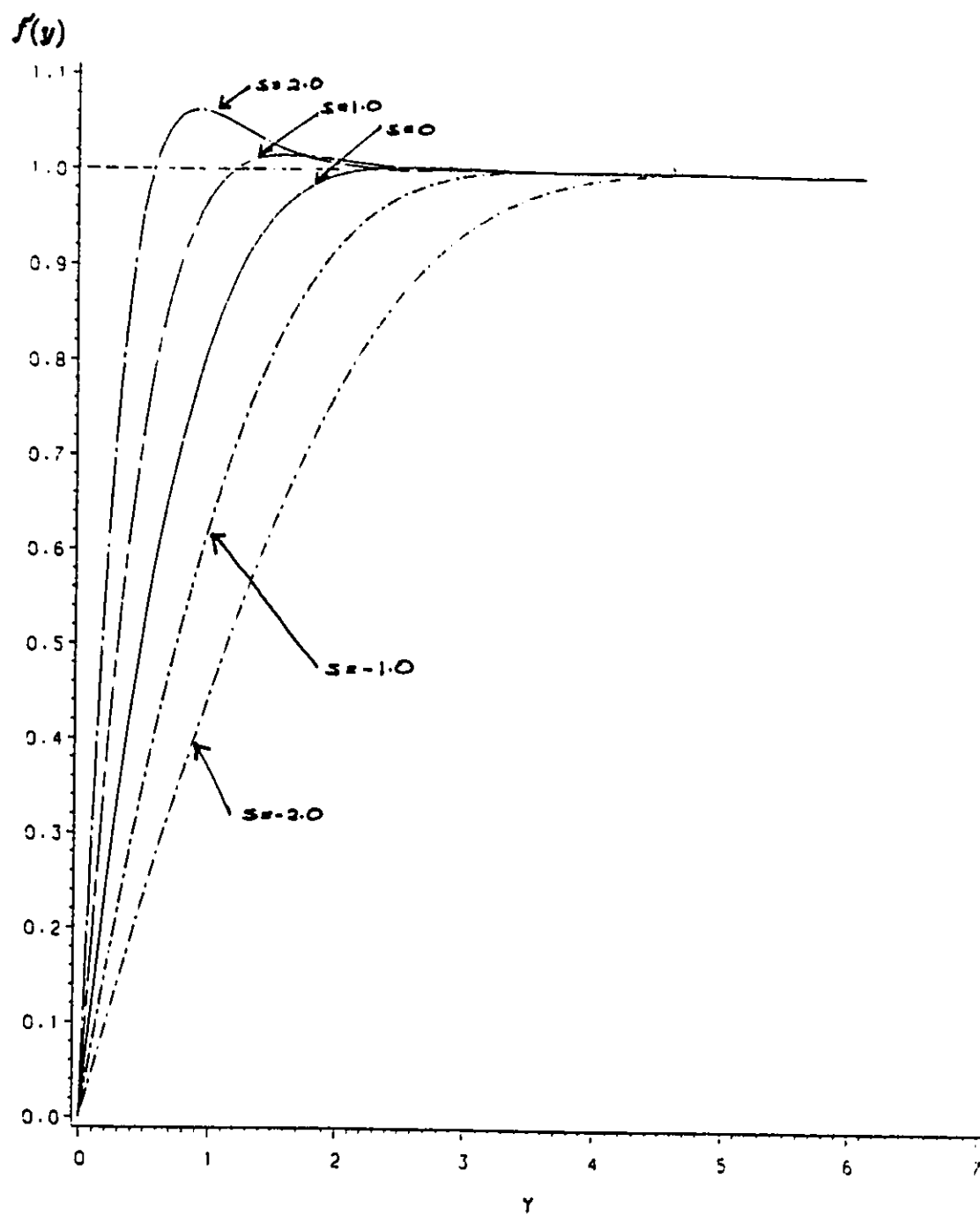


FIGURE 8.2: Velocity profiles f' for various values of s at $We = 0.1$.

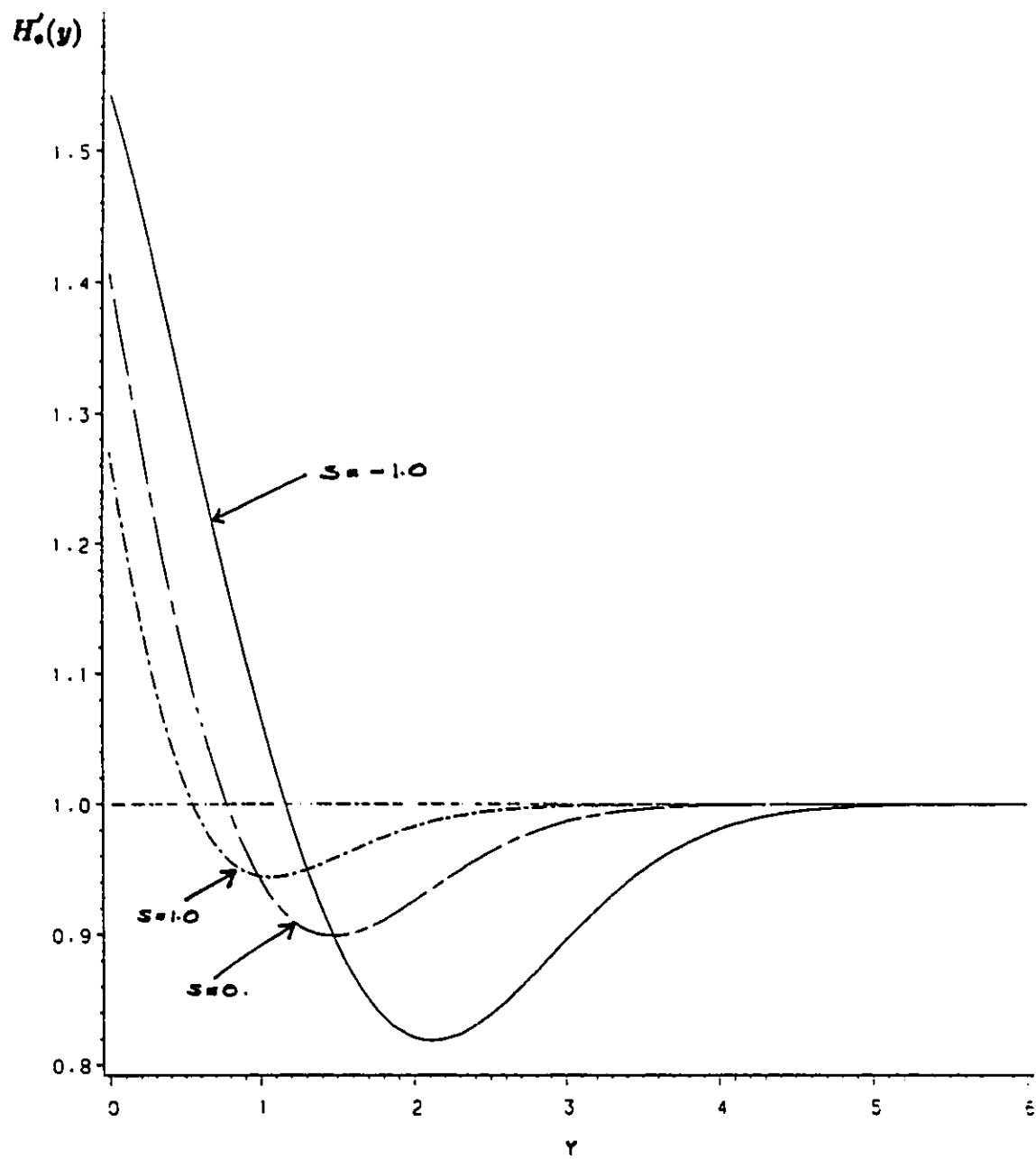


FIGURE 8.3: Velocity profiles H'_0 for different values of s_0 .

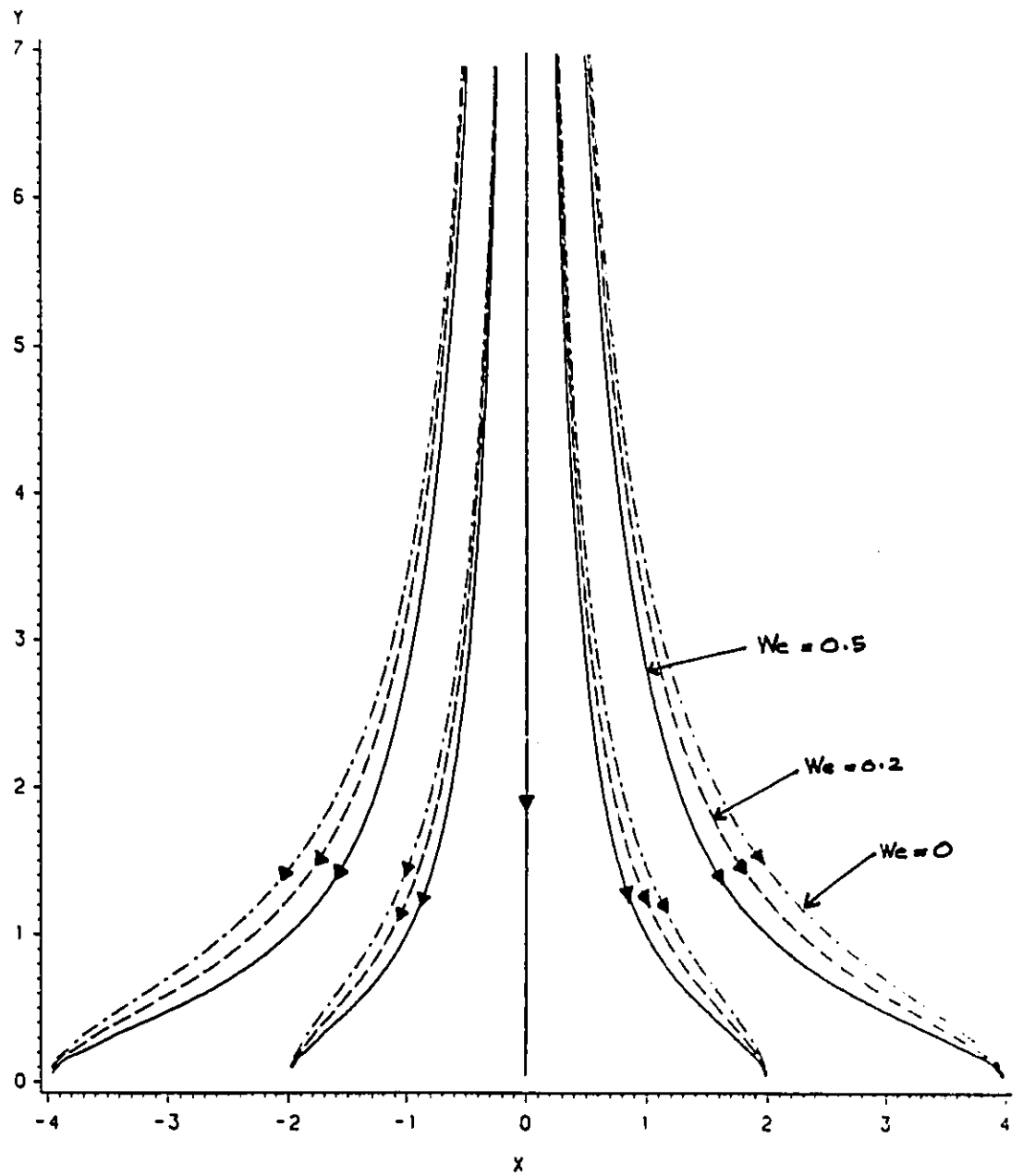


FIGURE 8.4: Streamline pattern for orthogonal flow with $k = 1$, $s = 1.0$ and $We = 0, 0.2$ and 0.5 .

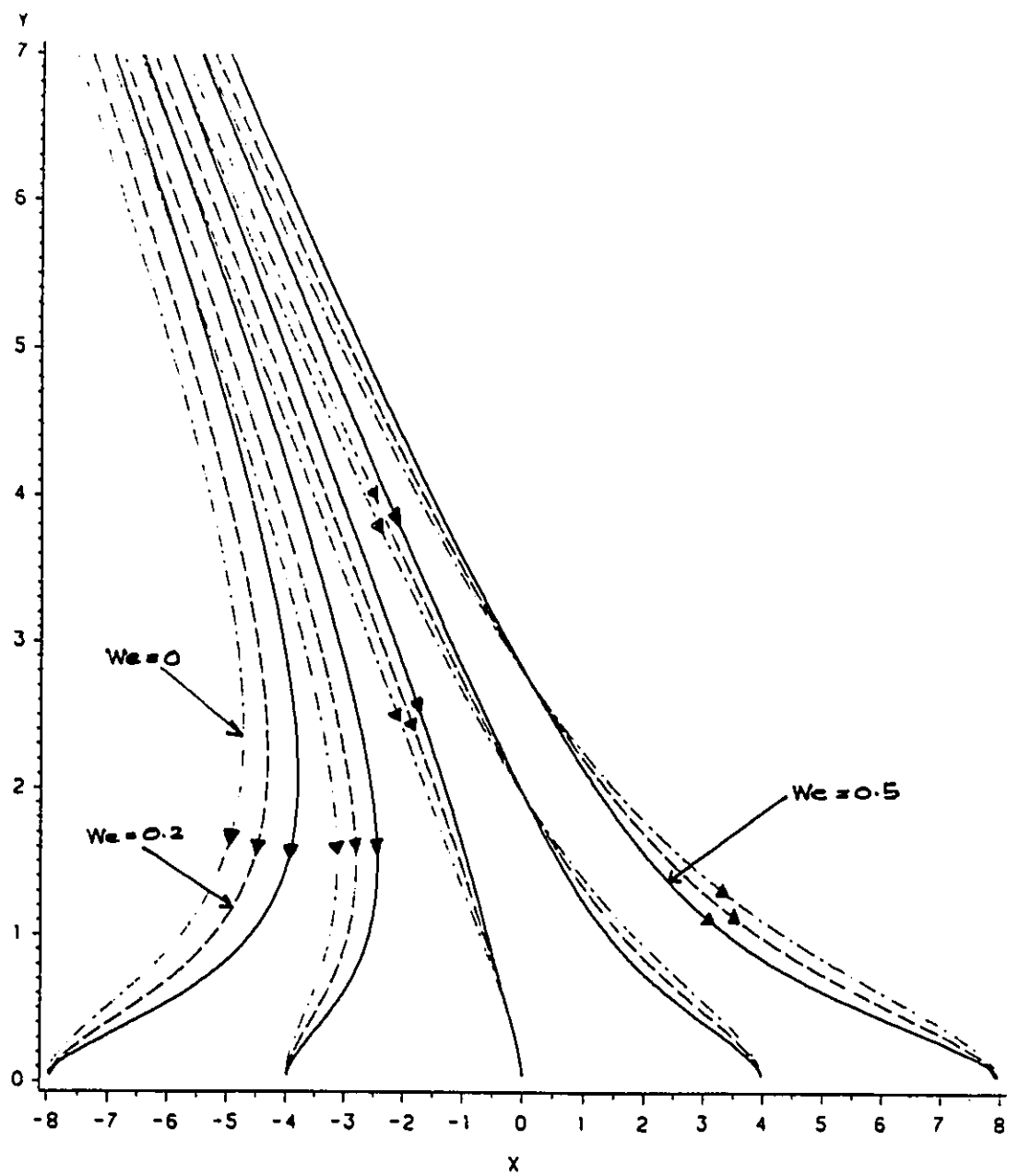


FIGURE 8.5: Streamline pattern for oblique flow with $k = 1$, $s = 1.0$ and $We = 0, 0.2$ and 0.5 .

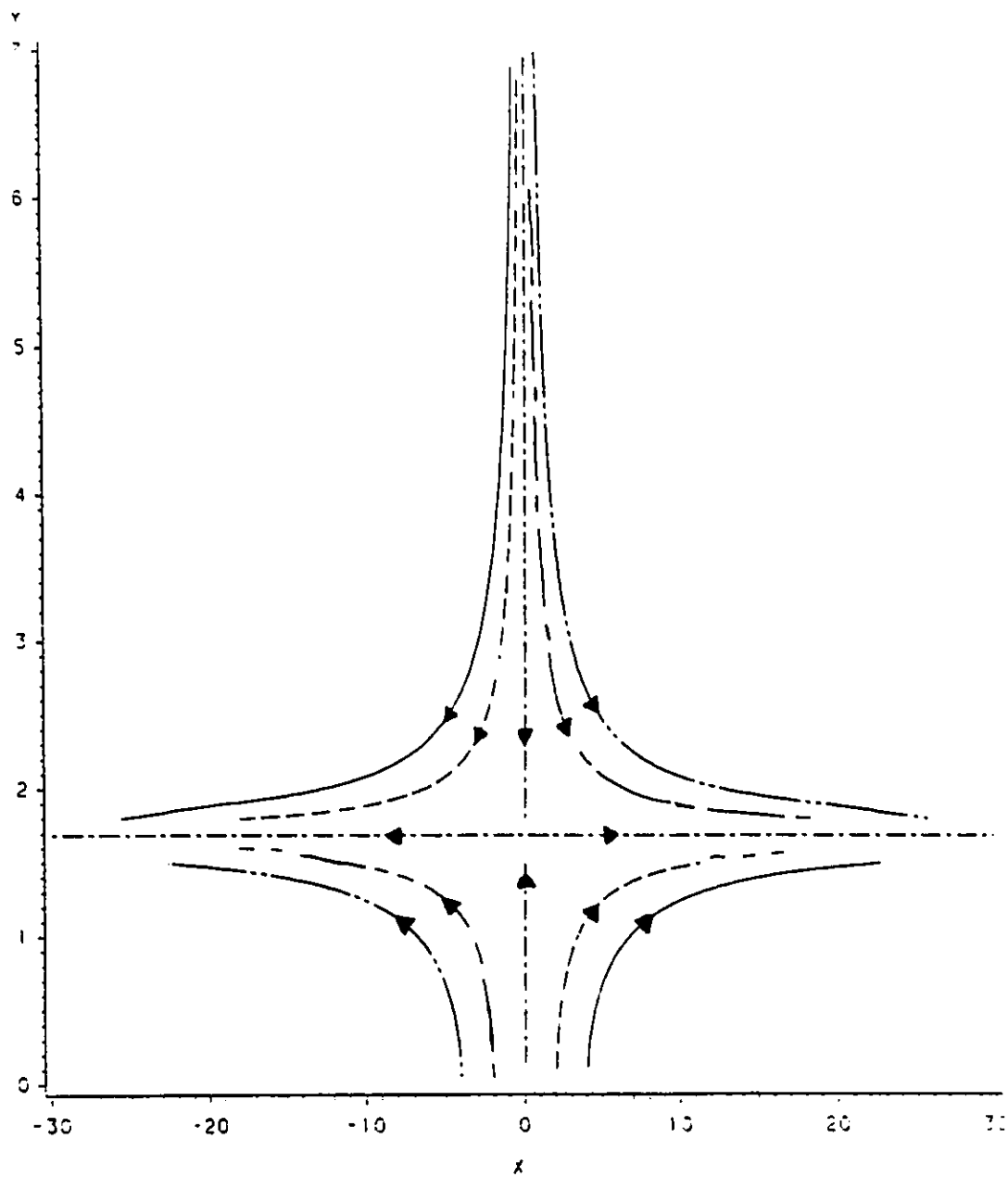


FIGURE 8.6: Streamline pattern for orthogonal flow with $k = 1$, $s = -1.0$ and $We = 0.2$.

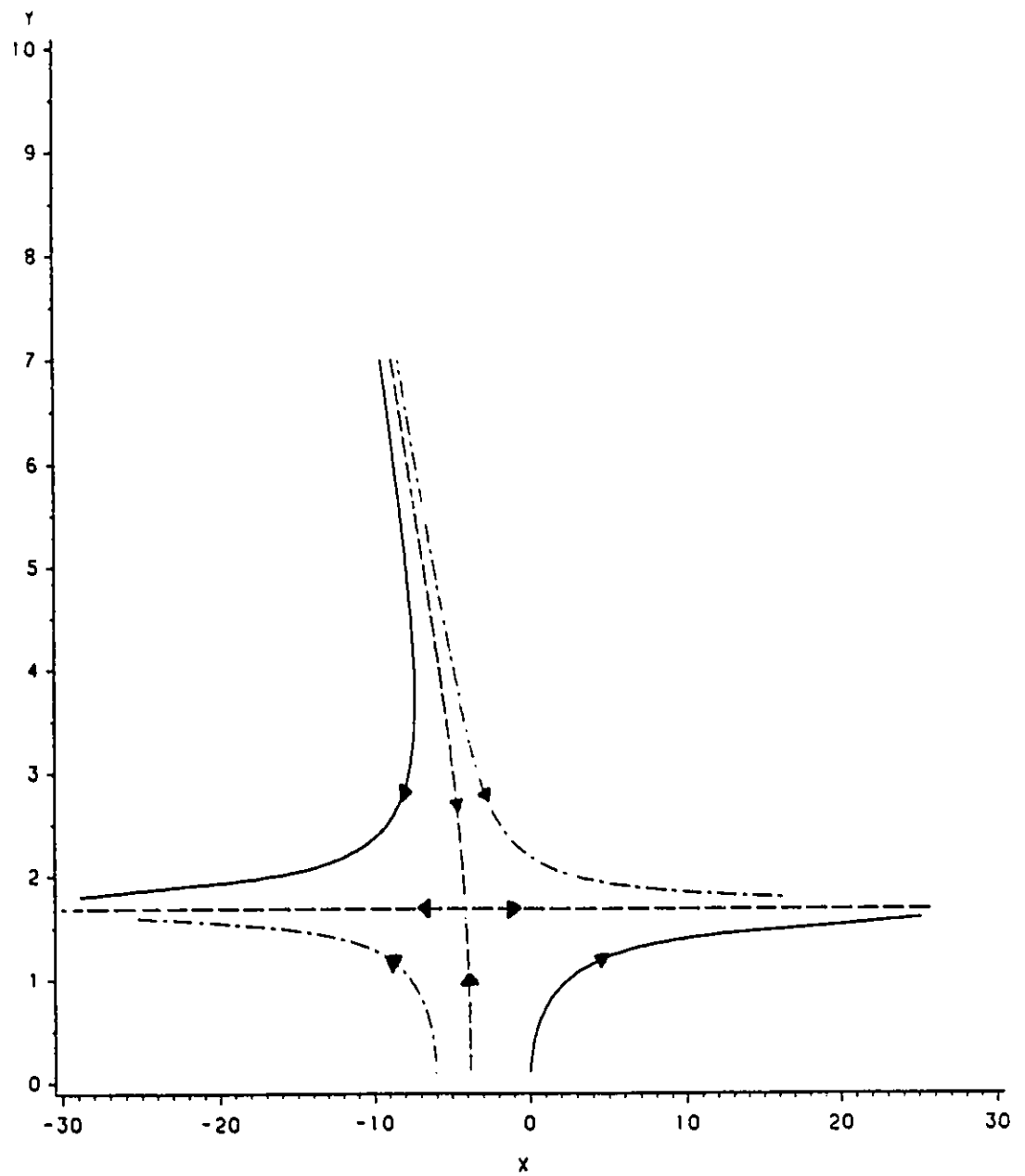


FIGURE 8.7: Streamline pattern for oblique flow with $k = 1$, $s = -1.0$ and $We = 0.2$.

CHAPTER 9

CONCLUSIONS AND SCOPE OF FUTURE WORK

9.1 THEORETICAL INVESTIGATION.

Steady plane flows of viscous incompressible and electrically conducting fluids in the presence of a magnetic field were studied under various assumptions. Both infinitely and finitely conducting fluids were considered. A new approach using generalized von Mises coordinates and other coordinate systems employing real and imaginary components of an analytic function of z was developed and used to obtain exact solutions of MHD aligned, MHD orthogonal and MHD variably-inclined fluid flows. The advantages of this method are:

- (i) it is far simpler than the existing inverse and semi-inverse methods, and
- (ii) it yields equations which do not require any additional assumptions to solve them.

Primarily, we posed and answered the following two questions: 1) Can a viscous fluid of constant viscosity flow along a given of curves $l(x, y) = \text{constant}$? 2) Given a family of streamlines in a viscous fluid flow, what are the exact solutions of the flow defined by the given streamline pattern?

To answer these two questions, we developed a new approach which is an extension of Martin's work [1971]. We initiate the investigation of the first question

by assuming that fluid flows along the given family of curves $l(x, y) = \text{constant}$ so that $\psi(x, y) = \text{constant}$ along these curves as well. Therefore, fluid flows along $l(x, y) = \text{constant}$ provided there exists some function $\gamma(\psi)$ such that

$$l(x, y) = \gamma(\psi)$$

Fluid flows along $l(x, y) = \text{constant}$ if the solutions obtained for $\gamma(\psi)$ is such that $\gamma'(\psi) \neq 0$.

This approach has been used to recover some existing exact solutions and yield several new exact solutions of finitely and infinitely conducting MHD aligned, infinitely conducting MHD orthogonal and infinitely conducting MHD variably-inclined fluid flows.

In the case of infinitely conducting MHD aligned flows, we showed that Hamel's problem has more solutions than the four well-known solutions of ordinary viscous fluid flow.

We are hoping to use this new approach more extensively in different areas of fluid mechanics to obtain new exact solutions of flow problems. Our aim is to be able to use this approach and its extensions to obtain exact solutions of physical problems when boundary conditions are also imposed.

In this part of the work, we also developed the bar-system for MHD transverse-aligned fluid flows and successfully studied some confluent flows. Dozens of confluent flows in MHD and other areas of fluid mechanics need to be studied and we are planning to do so in the near future. Many fluid flow processes can be realistically approximated by confluent flows and, therefore, their study has and will continue to provide some meaningful insights into the flow processes.

9.2 NUMERICAL INVESTIGATION.

We have studied the oblique flow of a viscoelastic fluid impinging on a flat porous wall with suction or blowing on the wall. In particular, we investigated the behaviour of the fluid near the wall for various magnitudes of suction or blowing. We found that when suction is applied the fluid penetrates the wall while blowing introduces an imaginary wall in the flow region and the fluid flows on both sides of this imaginary wall. It is also found that the position of this imaginary wall depends upon the magnitude of the blowing and upon the Weissenberg number. The process of suction and blowing has its importance in dealing with the manufacturing of polymers, ceramics, paints and other viscoelastic fluids. Blowing is used to add reactants, cool the surfaces, prevent corrosion or scaling and reduce the drag. Suction is applied to chemical processes to remove reactants.

In recent years, the movement towards consideration of complex flows of non-Newtonian fluids has been growing. These flows involve abrupt changes in geometry, often associated with free surface phenomena. Flow through commercial pipe fittings and pumps, die swell, fibre spinning and various other aspects of polymer processing are examples of industrially important situations which provide the practical motivation for this move towards complex flow problems. Although significant contributions have been made in this area, there is a general agreement that even the relatively simple non-Newtonian models used in existing flow situations are too complex to be employed in these situations. There are still a lot of problems to be solved in this area of fluid mechanics, among them is flow of different non-Newtonian fluids through channels or through abrupt contractions. We propose to study some of these problems in the coming years.

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